Extreme value distributions for open systems

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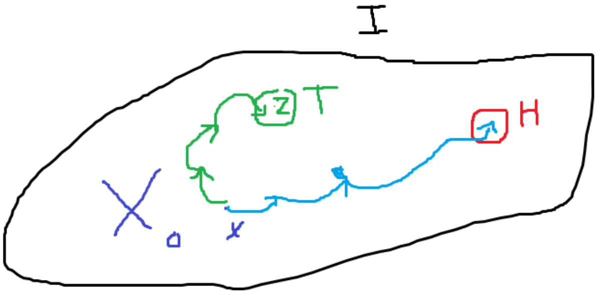
I will give a description of the results of the paper :

- *Targets and Holes*, with P Koltai and P Giulietti, arXiv:1909.09358

- This work is motivated by the appearance of extreme events in specific natural contexts. We are interested in the statistical description of phenomena where a perishable dynamics (i.e., an open system) is approaching a fixed target state.
- As examples one can think of the process describing a hurricane approaching a city or a pandemic outbreak (with the underlying space being the spatial distribution) approaching a critical extension, before they disappear.
- Thus, the dynamical setting is novel in that it has two main features: in the phase space, on one hand there is a target point which will be approximated by small balls around it, and on the other hand there is an absorbing region which terminates the process on entering it.

Illustration



- A one dimensional prototype of such situation can be formulated as an *extreme value problem for an open system*, thus allowing a rigorous study. Similar setups, restricted to the presence of shrinking targets or absorbing regions, but not both, have already been studied in many situations.
- We consider a dynamical system where there is an absorbing region, a *hole* H, such that an orbit entering terminates its evolution (i.e., it is lost forever).
- By considering the orbits of the whole state space, it is possible to construct a *surviving set*. On this we fix a point and a small ball around it, the *target set* B. We investigate the probability of hitting B for the first time after n steps while avoiding H, in the $n \to \infty$ limit. We will show that this question can be formulated in a precise probabilistic manner by introducing conditionally invariant probability measures for the open system.

- We consider Lasota–Yorke maps, i.e., uniformly expanding maps, $\inf_I |T'| = \beta > 1$, such that there exists a finite partition of the interval I with the property that T restricted to the closure of each element is C^1 and monotone, $T : I$ on the unit interval I and a transfer operator with a potential g of bounded variation (BV).
- We denote with $\mathcal{L}$ the transfer (Perron–Frobenius) operator associated to T and g; it acts on functions $f \in BV \cap L^1(\mu_g)$ as

$$\mathcal{L}f(x) = \sum_{Ty=x} f(y)g(y), \tag{1}$$

where μ_g is the conformal measure left invariant by the dual $\mathcal{L}^*$ of the transfer operator,

$$\mathcal{L}^* \mu_g = e^{P(g)} \mu_g,$$

where $P(g)$ is the topological pressure of the potential g.

- For simplicity, we will restrict ourselves to the potential $g = \frac{1}{|T'|}$. First of all note that, in this case, the conformal measure $\mu_{|T'|^{-1}}$ will be Lebesgue (denoted by m) and $P(g) = 0$.
- We then consider a proper subset $H \subset I$ of measure $0 < m(H) < 1$, called the *hole*, and its complementary set $X_0 = I \setminus H$. We denote by $X_n = \bigcap_{i=0}^{n} T^{-i} X_0$ the set of points that have not yet fallen into the hole at time n. The *surviving set* will be denoted by $X_\infty = \bigcap_{n=1}^{\infty} X_n$. The key object in our study are conditionally invariant probability measures.

- A probability measure ν which is absolutely continuous with respect to Lebesgue is called a conditionally invariant probability measure if it satisfies for any Borel set $A \subset I$ and for all $n > 0$ that

$$\nu(T^{-n}A \cap X_n) = \nu(A)\ \nu(X_n). \tag{2}$$

  We use for it the abbreviation a.c.c.i.p.m.
  The measure ν is supported on X_0, $\nu(X_0) = 1$, and moreover

$$\nu(X_n) = \alpha^n, \text{ where } \nu(X_1) = \nu(T^{-1}X_0) = \alpha < 1.$$

- We now introduce our first perturbed transfer operator defined on bounded variation function f as

$$\mathcal{L}_0(f) = \mathcal{L}(f\mathbf{1}_{X_0}). \tag{3}$$

We will use the following facts:

- Let $\nu = \mathbf{1}_{X_0} h_0 m$ with $h_0 \in L^1$ then ν is an a.c.c.i.p.m. if and only if $\mathcal{L}_0 h_0 = \alpha h_0$, for some $\alpha \in (0,1]$.
- Let α, h_0 be as above. Moreover, let μ_0 be a probability measure on I such that $\mathcal{L}_0^* \mu_0 = \alpha \mu_0$. Then μ_0 is supported in X_∞ and the measure Λ with

$$\Lambda = h_0\, \mu_0 \quad \text{is } T\text{-invariant.}$$

- For any $v \in L^1(\mu_0)$ and $w \in L^\infty(\mu_0)$ we have the duality relationship:

$$\int \mathcal{L}_0 v\ w\ d\mu_0 = \alpha \int v\ w \circ T\ d\mu_0. \tag{4}$$

We are now strengthening our assumptions by taking small holes since that will allow us to apply the spectral approach of extreme value theory.

- For each $\chi \in (\beta, 1)$ there exists $a, b > 0$, independent of H, such that, for each w of bounded variation:

$$\|\mathcal{L}^n w\|_{BV} \leq a\chi^n \|w\|_{BV} + b|w|_1 \tag{5}$$

$$\|\mathcal{L}_0^n w\|_{BV} \leq a\chi^n \|w\|_{BV} + b|w|_1. \tag{6}$$

- We introduce a so-called triple norm, defined by $\mathcal{P}_1 := \sup_{\|w\|_{BV}\leq 1} |\mathcal{P}w|_1$, where $w \in \mathrm{BV}$ and the linear operator $\mathcal{P}$ maps into L^1. It is easily proven that

$$\mathcal{L} - \mathcal{L}_{01} \leq e^{P(g)} m(H) = m(H). \tag{7}$$

- The following result is proved in [Liverani-Maume]. For each $\chi_1 \in (\chi, 1)$ and $\delta \in (0, 1-\chi_1)$, there exists $\epsilon_0 > 0$ such that if $\mathcal{L}_0 - \mathcal{L}_1 \leq \epsilon_0$ then the spectrum of $\mathcal{L}_0$ outside the disk $\{z \in \mathbb{C}, |z| \leq \chi_1\}$ is δ-close, with multiplicity, to the one of $\mathcal{L}$.
- We require that T has a unique invariant measure μ absolutely continuous with respect m with density h and moreover the system (I, T, μ) is mixing. Therefore $\mathcal{L}h = h$ and since $\mathcal{L}^* m = m$, we have that $\mu = hm$. Moreover, for any function v of bounded variation, there exists a linear operator $\mathcal{Q}$ with spectral radius $\mathrm{sp}(\mathcal{Q})$ strictly less than 1, such that

$$\mathcal{L}v = h \int v \, dm + \mathcal{Q}v. \tag{8}$$

- By the closeness of the spectra the same representation holds for $\mathcal{L}_0$, namely there will be a number λ_0, a non-negative function h_0 of bounded variation, a probability measure μ_0 and a linear operator $\mathcal{Q}_0$ with spectral radius strictly less than λ_0 such that for any $v \in \mathrm{BV}$:

$$\mathcal{L}_0 h_0 = \lambda_0 h_0, \;\; \mathcal{L}_0^* \mu_0 = \lambda_0 \mu_0 \tag{9}$$

$$\lambda_0^{-1} \mathcal{L}_0 v = h_0 \int v \, d\mu_0 + \mathcal{Q}_0 v. \tag{10}$$

- For a fixed *target point* $z \in X_\infty$ let us consider the observable

$$\phi(x) = -\log|x-z| \quad \text{for } x \in I,$$

and the function

$$M_n(x) := \max\{\phi(x), \cdots, \phi(T^{n-1}x)\}.$$

For $u \in \mathbb{R}_+$, we are interested in the probabilities of $M_n \le u$, where M_n is now seen as a random variable on a suitable (yet to be chosen) probability space $(\Omega, \mathbb{P})$. First of all we notice that the set of $x \in I$ for which it holds $\{M_n \le u\}$ is equivalent to the set $\{\phi \le u, \ldots, \phi \circ T^{n-1} \le u\}$. In turn this is the set $E_n := (B^c \cap T^{-1}B^c \cdots \cap T^{-(n-1)}B^c)$ where, for simplicity of notation, we denote with B^c the complement of the open ball $B := B(z, e^{-u})$, which we call the *target* (set).

- So far we are following points which will enter the ball B for the first time after at least n steps, but we should also guarantee that they have not fallen into the hole before entering the target. Therefore we should consider the event: $E_n \cap X_{n-1}$ conditioned on X_{n-1}, i.e., conditioned on the event of not terminating at least for $n-1$ steps. To assure that, the natural sequence of probability measures is given by the following

Definition

For any Borel set $A \subset I$ and any $n \geq 1$ we introduce the sequence of probability measures:

$$\mathbb{P}_n(A) := \frac{\nu(A \cap X_{n-1})}{\nu(X_{n-1})}.$$

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- Suppose now that, rather than taking one ball B, we consider a sequence of balls $B_n := B(z, e^{-u_n})$ centered at the target point z and of radius e^{-u_n}. Therefore:

$$\mathbb{P}_n(M_n \leq u_n) = \frac{1}{\nu(X_{n-1})} \int_I \mathbf{1}_{B_n^c \cap X_0}(x) \cdots \mathbf{1}_{B_n^c \cap X_0}(T^{n-1}x) d\nu, \tag{11}$$

  and we will consider the limit for $n \to \infty$, where u_n is a *boundary level* sequence which guarantees the existence of a non-degenerate limit.
- We anticipate that such a sequence will be dictated directly by the proof below and it must satisfy for a given τ

$$n\, \Lambda\big(B(z, e^{-u_n})\big) \to \tau \quad \text{as } n \to \infty. \tag{12}$$

By introducing our second perturbed operator $\tilde{\mathcal{L}}_n : BV \to BV$ acting as

$$\tilde{\mathcal{L}}_n v = \mathcal{L}_0(v\mathbf{1}_{B_n^c}) = \mathcal{L}(v\mathbf{1}_{B_n^c}\mathbf{1}_{X_0}),$$

it is straightforward to check that

$$\mathbb{P}_n(M_n \leq u_n) = \frac{1}{\alpha^{n-1}} \int_I \tilde{\mathcal{L}}_n^n h_0 \, dm. \tag{13}$$

Roughly speaking, when $n \to \infty$, the operator $\tilde{\mathcal{L}}_n$ converges to $\mathcal{L}_0$ in the spectral sense as $\mathbf{1}_{B_n^c}$ becomes less and less relevant in $\mathcal{L}_0(v\mathbf{1}_{B_n^c})$. In particular, the top eigenvalue of $\tilde{\mathcal{L}}_n$ will converge to that of $\mathcal{L}_0$ and this will allow us to control the asymptotic behavior of the integral on the right hand side of (13).

Assume that $h_- := \text{ess}\inf_{\mathrm{supp}(\Lambda)} h_0 > 0$ i.e. the essential infimum is taken with respect to Λ. Let

$$r_{k,n} := \frac{\Lambda(B_n \cap T^{-1}B_n^c \cap \cdots \cap T^{-k}B_n^c \cap T^{-(k+1)}B_n)}{\Lambda(B_n)},$$

where $r_{k,n}$ is the conditional probability with respect to Λ, that we return to B_n exactly after $k+1$ steps. Assume that

$$r_k = \lim_{n\to\infty} r_{k,n} \quad \text{exists for all } k.$$

Assume

- **A1.** The operators $\tilde{\mathcal{L}}_n$ enjoy the same Lasota–Yorke inequalities (5) with the same expansion constant χ and b in front of the weak norm.
- **A2**. We now compare the two operators; here the weak and strong Banach spaces will be again L^1 and BV. We have:

$$\int |(\mathcal{L}_0 - \tilde{\mathcal{L}}_n)v| \, dm = \int |\mathcal{L}_0(v\mathbf{1}_{B_n})| \, dm \leq \|v\|_{BV} \, m(B_n \cap X_0). \tag{14}$$

Then, for the triple norm, $\mathcal{L} - \tilde{\mathcal{L}}_{n1} \leq m(B_n \cap X_0)$ and therefore for n large enough we get the following spectral properties, analogous of (9), namely:

$$\tilde{\mathcal{L}}_n h_n = \lambda_n h_n, \;\; \tilde{\mathcal{L}}_n^* \mu_n = \lambda_n \mu_n \tag{15}$$

$$\lambda_n^{-1} \tilde{\mathcal{L}}_n g = h_n \int g \, d\mu_n + \widetilde{\mathcal{Q}}_n g, \tag{16}$$

where $h_n \in \mathrm{BV}$, μ_n is a Borel measure and $\widetilde{\mathcal{Q}}_n$ a linear operator with spectral radius less than one; moreover $\sup_n \mathrm{sp}(\widetilde{\mathcal{Q}}_n) < \mathrm{sp}(\mathcal{Q}) < 1$.

- **A3.** Next, we need to show that

$$\sup\left\{\int(\mathcal{L}_0-\tilde{\mathcal{L}}_n)v\,d\mu_0 : v\in \mathrm{BV}, \|v\|_{\mathrm{BV}}\le 1\right\}\times\|\mathcal{L}_0(h_0\mathbf{1}_{B_n})\|_{\mathrm{BV}}\le C_\sharp\Delta_n, \tag{17}$$

where

$$\Delta_n := \int\mathcal{L}_0(\mathbf{1}_{B_n}h_0)\,d\mu_0=\alpha\Lambda(B_n)$$

and $C_\sharp$ is a constant. Notice that the first term on the left hand side of (17) is the triple norm $\mathcal{L}_0-\tilde{\mathcal{L}}_{n\mu_0}$. This is bounded by $\alpha\mu_0(B_n)$. The second factor is bounded by the Lasota–Yorke inequality with a constant C_{h_0} depending on h_0. Then by the first standing assumption $\alpha C_{h_0}\mu_0(B_n)\le\frac{\alpha C_{h_0}}{h_-}\Lambda(B_n)$.

- **A4.** We now define the following quantity for $k \geq 0$:

$$q_{k,n} := \frac{\int (\mathcal{L}_0 - \tilde{\mathcal{L}}_n)\tilde{\mathcal{L}}_n^k(\mathcal{L}_0 - \tilde{\mathcal{L}}_n)(h_0)\, d\mu_0}{\Delta_n}. \tag{18}$$

By the duality properties enjoyed by the transfer operators with respect to our standing assumption, it is easy to show that

$$q_{k,n} = \alpha^{k+1} r_{k,n}. \tag{19}$$

We observe that by the Poincaré Recurrence Theorem with respect to the invariant measure Λ, as $r_{k,n}$ is the probability that the system returns to B_n in exactly $k+1$ steps, we have

$$\sum_{k=0}^{\infty} \alpha^{-(k+1)} q_{k,n} = \sum_{k=0}^{\infty} r_{k,n} = 1.$$

We denote by θ the *extremal index* (EI), which will be therefore between 0 and 1 :

$$\theta := 1 - \sum_{k=0}^{\infty} r_k.$$

- With our standing assumption, since we satisfy A1–A4, the perturbation theorem by Keller and Liverani gives
$$\lambda_n = \alpha - \theta\ \Delta_n + o(\Delta_n) = \alpha \exp\left(-\tfrac{\theta}{\alpha}\Delta_n + o(\Delta_n)\right), \text{ as } n \to \infty, \qquad (20)$$
or equivalently,
$$\lambda_n^n = \alpha^n \exp\left(-\tfrac{\theta}{\alpha} n\Delta_n + o(n\Delta_n)\right). \qquad (21)$$
- We now substitute (21) in the right hand side of (13) and use (15) to get
$$\mathbb{P}_n(M_n \leq u_n) = \frac{1}{\alpha^{n-1}} \int \lambda_n^n h_n\, dm \int h_0\, d\mu_n + \lambda_n^n \int \widetilde{\mathcal{Q}}_n^n h_0\, dm$$
$$= \alpha \exp(-\tfrac{\theta}{\alpha} n\Delta_n + o(n\Delta_n)) \int h_n\, dm \int h_0\, d\mu_n + \lambda_n^n \int \widetilde{\mathcal{Q}}_n^n h_0\, dm.$$
- It has been proved that $\int h_0\, d\mu_n \to 1$ for $n \to \infty$ and shown how to normalize h_n and μ_n in such a way that $\int h_n\, d\mu_0 = 1$. But in our case we have instead the term $\int h_n\, dm$. Now we observe that by the perturbative theorem of Liverani-Keller, we have that $|h_n - h_0|_1 \to 0$ as $n \to \infty$. Moreover
$$\int h_0\, dm = \frac{1}{\alpha} \int \mathcal{L}_0 h_0\, dm = \frac{1}{\alpha} \int \mathcal{L}(h_0 \mathbf{1}_{X_0})\, dm = \frac{1}{\alpha} \int h_0 \mathbf{1}_{X_0}\, dm = \frac{1}{\alpha}\nu(X_0) = \frac{1}{\alpha},$$
and this term will compensate the α in the numerator in the equality above.

- Note that the choice given by (12) is equivalent to $n\Delta_n \to \alpha\tau$. In this case λ_n^n will be simply bounded in n and $\int |\widetilde{\mathcal{Q}}_n^n(h_0)|\, dm \leq \mathrm{sp}(\mathcal{Q})^n \|h_0\|_{BV} \to 0$.
- In conclusion we have

$$\lim_{n\to\infty} \mathbb{P}_n(M_n \leq u_n) = e^{-\tau\theta}, \tag{22}$$

  which is the Gumbel's law.
- Suppose that all the iterates $T^n, n \geq 1$ are continuous at z and also that h_0 is continuous at z when the latter is a periodic point. Then we have:
  - If z is not a periodic point:

$$\mathbb{P}_n(M_n \leq u_n) \to e^{-\tau}.$$

  - If z is a periodic point of minimal period p, then

$$\mathbb{P}_n(M_n \leq u_n) \to e^{-\tau\theta},$$

    where the extremal index θ is given by:

$$\theta = 1 - \frac{1}{\alpha^p |(T^p)'|(z)}$$

  Note that in literature, the *escape rate* η for our open system is usually defined as $\eta = -\log \alpha$ thus we can see the extremal index as

$$\theta = 1 - \frac{1}{e^{-p\eta}|(T^p)'|(z)}.$$

The dimension of the surviving set I

- Whenever the map T has large images and large images with respect to the hole H, then for all $z \in X_\infty$, there exists $t_0 > 0$ such that

$$\liminf_{n\to\infty} \frac{\log \mu_0(B(z, e^{-u_n}))}{\log e^{-u_n}} \geq t_0$$

and the Hausdorff dimension of the surviving set $HD(X_\infty)$ verifies

$$HD(X_\infty) \geq t_0.$$

- Therefore $u_n \leq -\frac{\log \tau}{t_0 - 2\delta} + \frac{\log n}{t_0 - 2\delta}$, which can also be written as

$$\sup_n \left\{ u_n - \frac{\log n}{t_0} \right\} \leq -\frac{\log \tau}{t_0}. \qquad (23)$$

- In the computational approach to extreme value theory, the boundary level u_n are chosen with the help of an affine function:

$$u_n = \frac{\log \tau^{-1}}{a_n} + b_n.$$

- The sequences a_n and b_n can be obtained with the help of the Generalized Extreme Value (GEV) distribution in order to fit Gumbel's law. The inequality in the previous page suggests that for n large $a_n \sim t_0$ and $b_n \sim \frac{\log n}{t_0}$, therefore we could attain a lower bound for the Hausdorff dimension of the surviving set. We can use the GEV distribution to estimate the sequences a_n, b_n, and we will show in future studies how to use such estimates to approach $HD(X_\infty)$.

Outside the surviving set

- Whenever we take the point $z \in X_\infty$ and by a suitable choice of the sequence u_n, we get a non-degenerate limit for our EVD, in particular different from 1. Instead, if we pick the point z outside the surviving set and no matter what the sequence u_n is, provided it goes to infinity, we get a degenerate limit equal to one for the EVD.
- Trivially that states that if the target point is off the surviving set, then the trajectories will not be able to approach it arbitrary close.
- First, we observe that the limit $\mathbb{P}_n(M_n \le u_n) \to 1,\ n \to \infty$ holds for any sequence u_n going to infinity, and for simplicity we now put $u_n = \log n$. Then we could reasonably argue that for the smallest $\hat{n}$ for which

$$\mathbb{P}_{\hat{n}}(M_{\hat{n}} \le \log \hat{n}) \sim 1,$$

then

$$\text{dist}(z, X_\infty) \sim \frac{1}{\hat{n}}.$$