Maximum Entropy Production hypothesis in conceptual climate models and stochastic sub-grid modelling for lattice-gas (and turbulent flows?)

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13/02/2020

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Motivations

Climate as a statistical physics problem

Climate = coarse-graining of meteorological variables (and others), usually on the radiative forcing space and time scales.

We know only the meteorological ($\simeq$ microscopic) laws, but we cannot resolve them explicitly due to computational cost.

Fundamental question : Evolution of the coarse-grained ($\simeq$macroscopic) variables ?

$\Rightarrow$ Sub-grid modelling $\Leftrightarrow$ Nonequilibrium statistical physics problem.

Particular case : turbulence modelling

A simple view of turbulence modelling [Pope, 2001] :

Causes : chaos (or complicated dynamics) + coarse-graining (finite resolution)
$\Rightarrow$ Unpredictability (in a deterministic sense)
$\Rightarrow$ Stochastic description (since Reynolds decomposition).

Consequence : enhanced mixing compared to diffusion. **State of the art** (not exhaustive) :

1877 : Eddy-viscosity (Boussinesq) ;

1963 : Heuristic dependence of eddy viscosity on grid size [Smagorinsky, 1963] ;

1972 : $k - \epsilon$ model focuses on the mechanisms that affect the turbulent kinetic energy [Jones and Launder, 1972] ;

1991 : Dynamic LES model [Germano et al., 1991].

Drawbacks : ad-hoc parameters not easy to calibrate in the context of climate modelling, no real description of the coarse-graining procedure in these models (except for LES), no description of fluctuations.

Goals

Part 1 : Construct simple climate models with less parametrizations (or no) for the turbulence.
$\Rightarrow$ Phenomenological approach to represent the enhanced turbulent mixing : the Maximum Entropy Production hypothesis.

Part 2 : What is the evolution law for coarse-grained variables for simple systems ?
$\Rightarrow$ Numerical analysis and modelling approach.

The Maximum Entropy Production hypothesis for climate modelling

Energy Balance models and Maximum Entropy Production

Energy Balance model : [BUDYKO, 1969, Sellers, 1969, North et al., 1981]

$$\partial_t e + \partial_\alpha F_\alpha = \mathcal{R}[T].$$

e : energy ; T temperature

F : energy fluxes ;

$\mathcal{R}[T]$: forcing (radiative budget).

Closure $F(e, \partial_\beta e)$?

Maximum Entropy Production (MEP) : closure hypothesis [Martyushev and Seleznev, 2006]

$$\max_{T(r,t)} \left\{ \sigma[T] = \int_{t=0}^{\mathcal{T}} \int_{r \in V} \frac{-\partial_\alpha F_\alpha}{T} \, \mathrm{d}r \, \mathrm{d}t \right\}$$

with

$$\int_{r \in V} \frac{\partial_\alpha F_\alpha}{T} \, \mathrm{d}r = 0 \quad \forall t \in [0 : \mathcal{T}]$$

and other constraints

$\Rightarrow$ Turbulence acts as a strong mixing of energy.

MEP used in a stationary context, for the meridional heat transport [Paltridge, , Pujol and Llebot, 2000, Herbert et al., 2011].

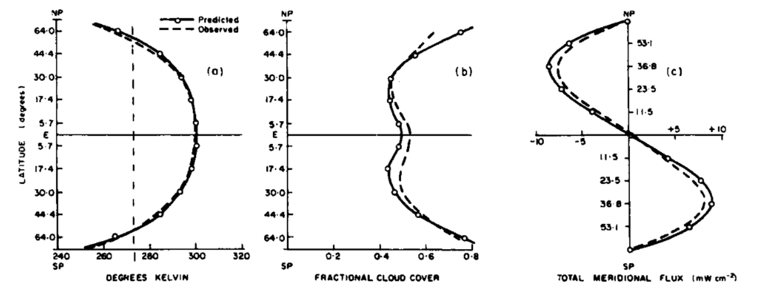


Figure 1 – Meridional profiles of temperature, cloud cover, and heat flux obtained by [Paltridge,].

Also for "over-parametrized" processes (hydrological cycle) (see [Kleidon, 2009]).

- On the vertical ? ⇒ Some dynamical "constraints" matter.
- MEP for time varying problems ?

$$\partial_t e + \partial_\alpha F_\alpha = \mathcal{R}$$

Radiative budget :

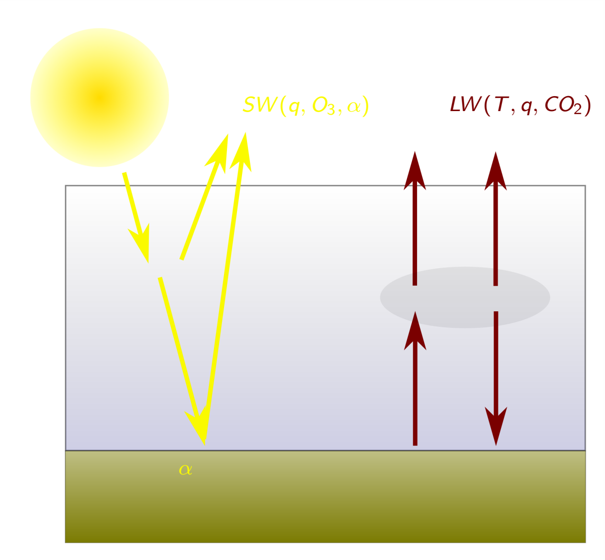
$$\mathcal{R} = SW + LW.$$

$=$ net energy input per unit time due to radiation.

Convective budget :

$-\partial_\alpha F_\alpha =$ convergence of energy due to (turbulent) fluid's motions.

Impacted by gravity.



Atmospheric convection

$$\partial_t e + \partial_\alpha F_\alpha = \mathcal{R}$$

Radiative budget :

$$\mathcal{R} = SW + LW.$$

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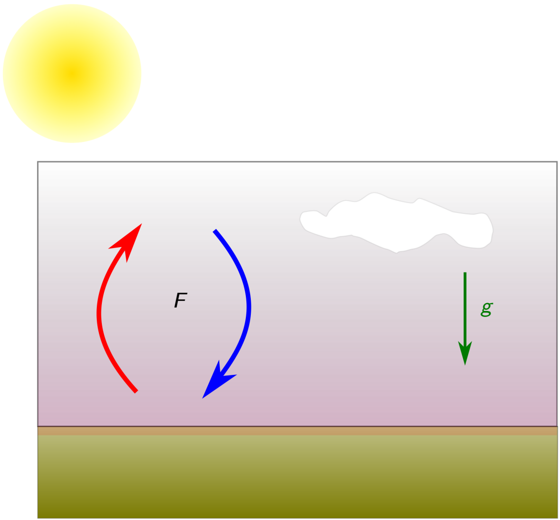


Our model

Heat transport is two steps :

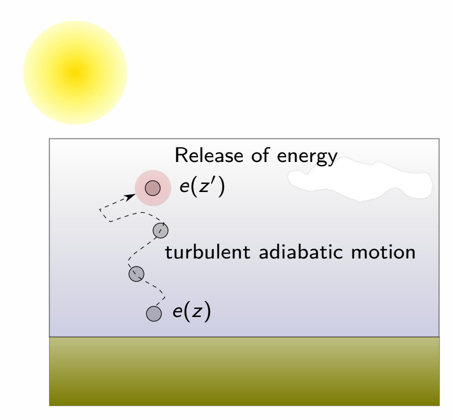
1. Adiabatic motion of fluid parcels with specific energy :

$$e = \underbrace{C_p T}_{\text{sensible heat}} + \underbrace{gz}_{\text{geopotential}} + \underbrace{Lq}_{\text{latent heat}}$$

2. The fluid parcel releases its energy at elevation z'.

Hypothesis : perfect gas, hydrostatic is used to compute the geopotential with the temperature profile.

Mass mixing rate : $m(z \to z')$.
$=$ mass of fluid parcels going from z to z' per unit time.



Our model

In stationary state :

$$\frac{\mathrm{d}M(z)}{\mathrm{d}t} = \sum_{z'} \left[m(z' \to z) - m(z \to z')\right] = 0,$$

$$\frac{\mathrm{d}E(z)}{\mathrm{d}t} = \sum_{z'} \left[F(z' \to z) - F(z \to z')\right] = 0$$

with $F(z \to z') = e(z)m(z \to z')$.

$m \geq 0$ and depends on the turbulent dynamics
$\Rightarrow$ usually parametrized.

We choose m (or F) that maximize the entropy production

$$\sigma = \sum_{z} \frac{\sum_{z'} [F(z' \to z) - F(z \to z')]}{T(z)},$$

taking into accout

$$e(z) = C_p T(z) + gz + Lq_s(T(z))$$

and

$$m(z) \geq 0.$$

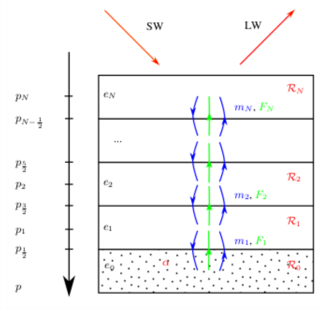
$\Rightarrow$ MEP with a minimal description of the energy fluxes (dynamics)

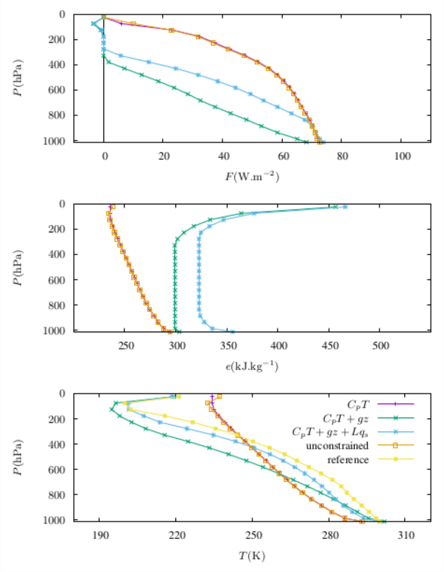
We can consider different mass scheme transport (which can be represented by a graph).

O_2 and q fixed according to [A. McClatchey et al., 1972]'s measurements for Tropics.

Radiative budget computed using the code of [Herbert et al., 2013].

Linear graph for the mass exchange.

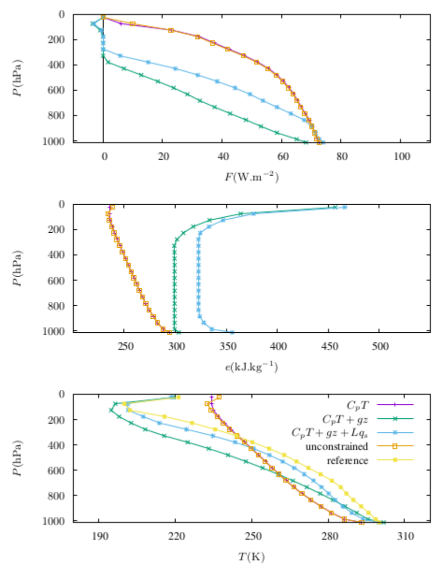


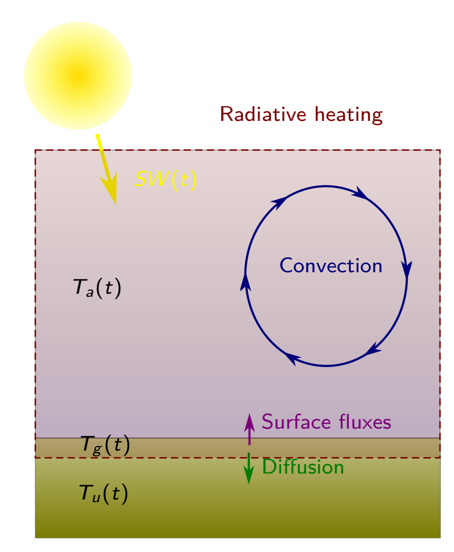


The MEP state depends on the energy description (and constraints).

- **No constraint on m** : unphysical heat exchange + overestimation of the vertical heat flux.
- $\boldsymbol{e = C_p T}$: overestimation of the vertical heat flux.
- $\boldsymbol{e = C_p T + gz}$: more realistic adiabatic gradient for the troposphere + stratification.
- $\boldsymbol{e = C_p T + gz + Lq_s}$: enhanced vertical energy flux.

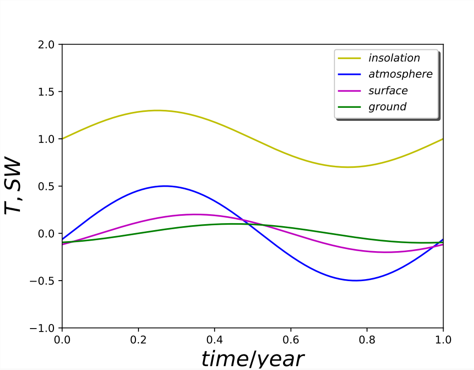
$\Rightarrow$ Simple but realistic results without parametrization for turbulence.





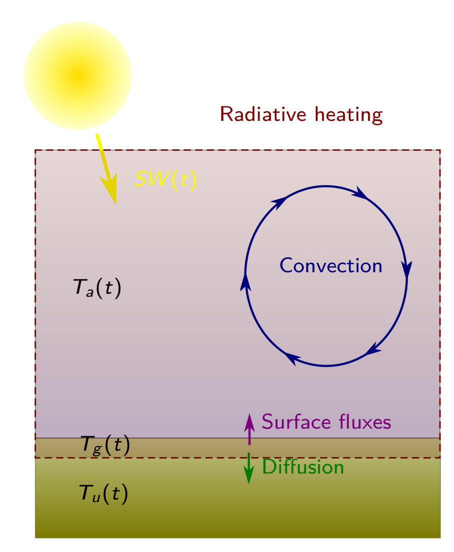
Temperatures' ranges and lags ?

[Stine et al., 2009, Stine and Huybers, 2012]



⇒ Conceptual (MEP) model approach.

Seasonal cycle



Not everything should be represented with MEP [Paillard and Herbert, 2013].

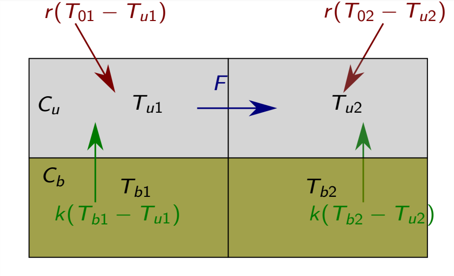
Fast (turbulent) processes : convection and surface fluxes.

Slow processes : radiative heating and diffusion in ground.

Our model

Forcing :

$$T_{01}(t) = 300 + 10\sin\left(\frac{2\pi}{\tau}t\right)\ K,$$
$$T_{02}(t) = 300 - 10\sin\left(\frac{2\pi}{\tau}t\right)\ K.$$



Control parameters :

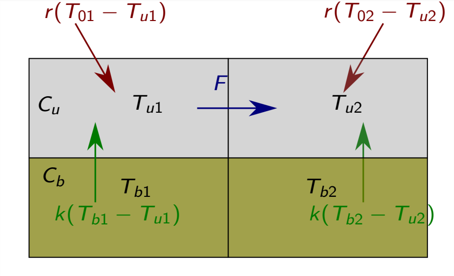
$$\mathcal{N}_b = \frac{C_b}{k\tau} \quad = \text{heating time of bottom boxes by diffusion},$$
$$\mathcal{N}_k = \frac{C_u}{k\tau} \quad = \text{heating time of top boxes by diffusion},$$
$$\mathcal{N}_r = \frac{C_u}{r\tau} \ll 1 \quad = \text{heating time of top boxes by "radiative heating"}.$$

Our model

$$\dot{T}_{u1} = \underbrace{\frac{1}{\mathcal{N}_r}(T_{01} - T_{u1}) + \frac{1}{\mathcal{N}_k}(T_{b1} - T_{u1})}_{\mathcal{F}_1} - Q,$$

$$\dot{T}_{u2} = \underbrace{\frac{1}{\mathcal{N}_r}(T_{02} - T_{u2}) + \frac{1}{\mathcal{N}_k}(T_{b2} - T_{u2})}_{\mathcal{F}_2} + Q,$$

$$\dot{T}_{bi} = -\frac{1}{\mathcal{N}_b}(T_{bi} - T_{ui}),$$



with $Q = \frac{F\tau}{C_u}$ fixed by MEP :

$$\max_{\{T_{u1}(t), T_{u2}(t)\}} \left\{ \sigma = \int_{t=0}^{1} \left(\frac{\dot{T}_{u1} - \mathcal{F}_1}{T_{u1}} - \frac{\dot{T}_{u2} - \mathcal{F}_2}{T_{u2}} \right) \mathrm{d}t \;\middle|\; \dot{T}_{u1} + \dot{T}_{u2} - \mathcal{F}_1 - \mathcal{F}_2 = 0 \ \forall t \right\}$$

Resolution : derivation of the dynamical equations and Newton's method.

Results

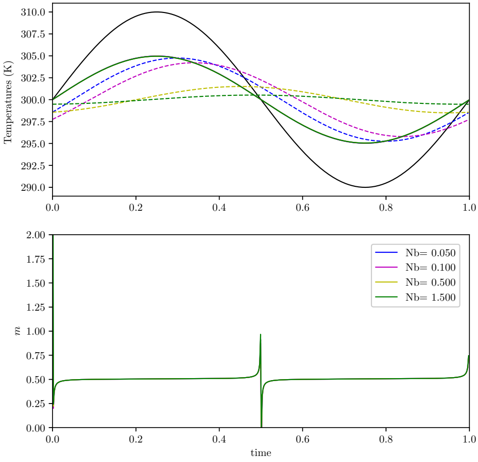
Influence of $\mathcal{N}_b$:
Lag of the ground with respect to the upper boxes.

Influence of $\mathcal{N}_r$:
Lag and amplitude of the top with respect to the forcing.

Influence of $\mathcal{N}_k$:
Lag and amplitude of the top with respect to the bottom.

$\Rightarrow$ Simple but realistic results without parametrization for turbulence.

Have to be tested on more realistic (meridional) models.



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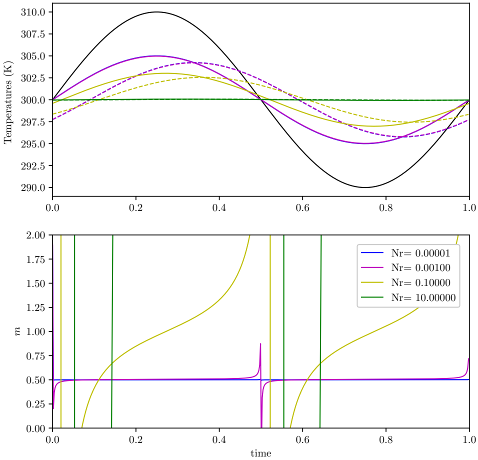

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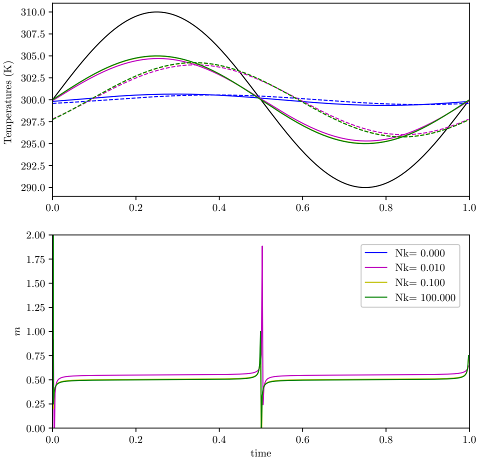


Conclusion

No new results, but comprehensive models with less adjustable parameters.

MEP hypothesis : convective mixing of the atmosphere is fast before the change in radiative budget and/or diffusion ($\simeq$ quasi-static approximation ? Scale' separation ?). It is not a "fundamental law".

$$\text{Convective time scale} \simeq \frac{Earth' size}{Wind' speed} \simeq \frac{10^7 \ m}{10 \ m.s^{-1}} = 10^6 \ s \simeq 0.03 \text{ years.}$$

How the convective energy mixing is done ? Dynamical "constraints" have to be considered (mass transport + energy terms).

$$\partial_t e = \underbrace{\gamma_{MEP}}_{\text{"fast" processes}} + \underbrace{\gamma_O}_{\text{"slow" processes}} + \underbrace{\mathcal{R}[T]}_{\text{Radiative forcing}}$$

Radiative forcing and slow processes (diffusion) are represented explicitly by usual laws.

Fast processes (turbulence) are fixed by MEP, taking into account relevant constraints :

$$\max \left\{ \sigma = \int_{t=0}^{\mathcal{T}} \int_{r \in V} \frac{\gamma_{MEP}}{T} \, \mathrm{d}r \, \mathrm{d}t \,\middle|\, \text{constraints} \right\}.$$

Perspectives

Long-term goal : Construct a minimal climate model with the lesser empirical parameters as possible.

- Seasonal cycle ;
- 3D with dynamical constraints (gravity and rotation) ;
- Water cycle.

Other interesting problems :

- Ice-albedo feedback ;
- Carbon cycle (paleoclimates) ;
- ...

Stochastic sub-grid model for a diffusive Lattice Gas (and extension to turbulent flows)

Nonequilibrium statistical physics : kinetic theory vs turbulence

Nonequilibrium statistical physics : *microscopic* $\underset{\text{coarse-graining}}{\rightarrow}$ *macroscopic*

Kinetic theory : *particles* $\underset{\text{coarse-graining}}{\rightarrow}$ *fluid particles*.

Turbulence modelling : *fluid particles* $\underset{\text{coarse-graining}}{\rightarrow}$ *mesh resolution*.

If Kinetic Theory is well established, why Turbulence Theory is not ?

1. Interaction between fluid particles ?
2. No clear separation of scales and simplifications (Boltzmann hypothesis).
   $\Rightarrow$ No "Local Thermodynamic Equilibrium" + important fluctuations.

Can we learn about turbulence from Kinetic Theory with no separation of scale ? (see [CHEN et al., 2004])

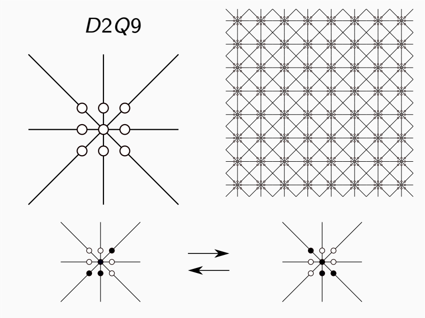
Lattice-Gas

Lattice-Gas = Cristallographic lattice $\mathcal{L}$ + Discrete sets of velocities and particles + Collisions and forcing rules

b channels at each node
$\rightarrow$ discrete velocities $\{c_i, i = 1, ..., b\}$

Boolean occupation $n_{*i}(x_*, t_*)$

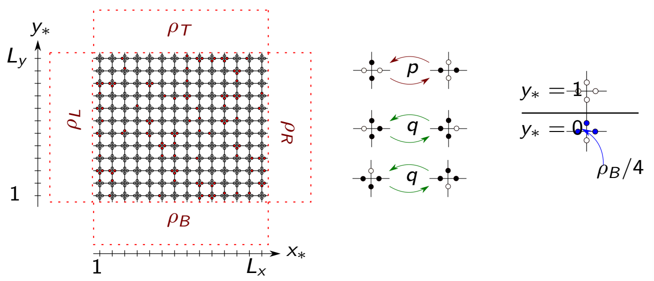
Local collision/forcing that conserve (or not) mass, impulsion, energy, ...



Discrete time microdynamics : Collision/Forcing and propagation

$$n_*(t_* + 1) = \mathcal{E}\ n_*(t_*) = \mathcal{P} \circ \mathcal{C}\ n_*(t_*).$$

$\simeq$ Discrete model of Kinetic Theory of gas [Hardy et al., 1973, Frisch et al., 1987, Grosfils et al., 1993, A Wolf-Gladrow, 2000, Rivet and Boon, 2005].



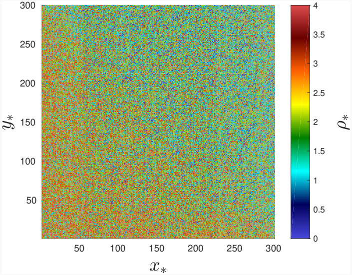
Microdynamics : Collision and propagation

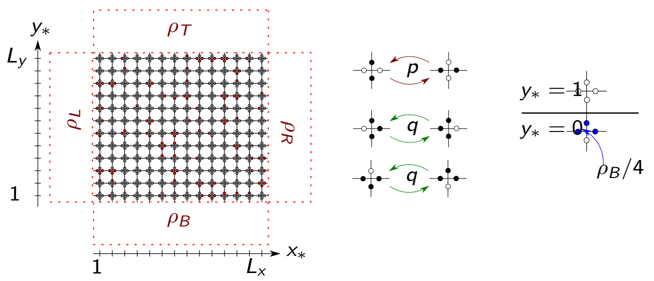
$$n_*(t_* + 1) = \mathcal{E}\ n_*(t_*) = \mathcal{P} \circ \mathcal{C}\ n_*(t_*).$$

Microscopic observables : density and current

$$\rho_*(r_*, t_*) = \sum_{i=1}^{4} n_{*i}(r_*, t_*),$$

$$j_{*\alpha}(r_*, t_*) = \sum_{i=1}^{4} c_{i\alpha} n_{*i}(r_*, t_*).$$





Ensemble average microdynamics :

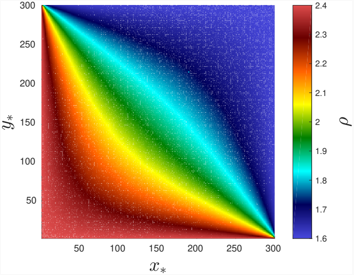
$$\epsilon = \frac{c}{L} \to 0, \qquad r = \epsilon r_*, \qquad t = \epsilon^2 t_*,$$

$$\rho = \langle \rho_* \rangle, \qquad j_\alpha = \langle j_{*\alpha} \rangle,$$

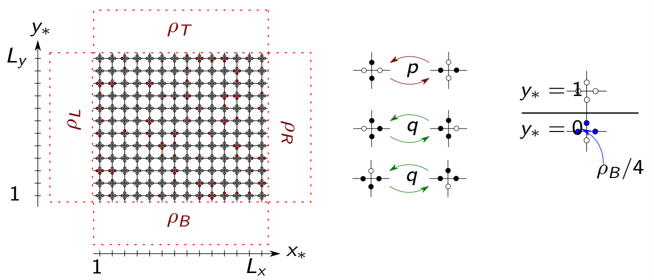
$$\partial_t \rho + \partial_\alpha j_\alpha = 0,$$

$$j_\alpha = \left(\frac{1}{4} - \frac{4}{q\rho^2} \right) \partial_\alpha \rho,$$

$$\langle (j_{*\alpha} - j_\alpha)^2 \rangle = \frac{\rho}{2} \left(1 - \frac{\rho}{4} \right) \equiv \sigma_*^2(\rho).$$



The microscopic model : the mesoscale ?

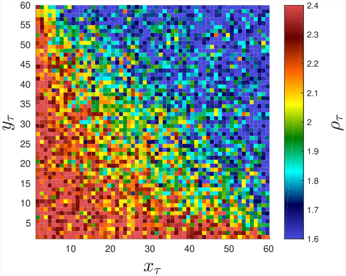


Coarse-grained dynamics :

$$1 \ll \tau \ll \infty, \qquad r_\tau = x_*/\tau, \qquad t_\tau = t_*/\tau,$$

$$q_\tau(r_\tau, t_\tau) = \frac{1}{\tau^3} \sum_{(r_*, t_*) \in \mathcal{M}(r_\tau, t_\tau)} q_*(r_*, t_*),$$

$$\partial_{t_\tau} \rho_\tau + \partial_{\alpha_\tau} j_{\tau\alpha} = 0,$$



What is the right closure for j_τ ?

The conditional probability distribution of the temporal variation of the coarse-grained current

Hypothesis :

1. Locality in space : response of j_τ depends only on j_τ, ρ_τ and $\nabla_\tau \rho_\tau$
2. Lag : j_τ don't instantaneously adjust to the local forcing (gradient of density) $\Rightarrow$ we model its temporal variation dj_τ.
3. Isotropy : $\mathrm{d}j_{\tau\alpha}$ depends on $j_{\tau\alpha}$, ρ_τ and $\partial_{\alpha_\tau} \rho_\tau$

$\Rightarrow$ We search the conditional probability distribution

$$p_\tau(\mathrm{d}j_{\tau\alpha} \mid j_{\tau\alpha}, \rho_\tau, \partial_{\alpha_\tau} \rho_\tau)$$

Reduced variable :

$$\delta_\tau(j, \rho, g) = \frac{\mathrm{d}j_\tau - \langle \mathrm{d}j_\tau \rangle_{j,\rho,g}}{\sqrt{\langle (\mathrm{d}j_\tau - \langle \mathrm{d}j_\tau \rangle_{j,\rho,g})^2 \rangle_{j,\rho,g}}}$$

where

$$\langle f \rangle_{j,\rho,g} \equiv \int f(v)\ p_\tau(v|j,\rho,g)\ \mathrm{d}v$$

is the conditional mean of any function f of $\mathrm{d}j_\tau$.

The conditional probability distribution of the temporal variation of the coarse-grained current

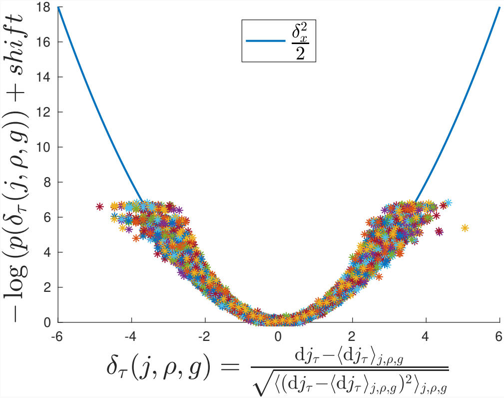
PDF of $\delta_\tau(j,\rho,g)$ for $\tau = 10, 20, 30$ and for all j, ρ, g have a universal normal behaviour.

$\Rightarrow$ Modelling the two first moments :

$$\langle \mathrm{d}j_\tau \rangle_{j,\rho,g},$$

$$\sqrt{\langle (\mathrm{d}j_\tau - \langle \mathrm{d}j_\tau \rangle_{j,\rho,g})^2 \rangle_{j,\rho,g}}.$$

as a function of τ, j, ρ and g



Sub-grid model : stochastic relaxation for the coarse-grained current

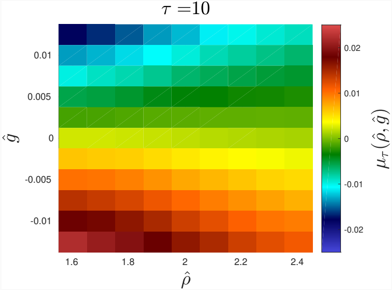
We assume a stochastic relaxation equation for the coarse-grained current :

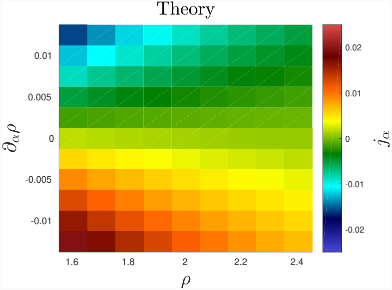
$$\mathrm{d}j_{\tau\alpha} = \underbrace{-\frac{j_{\tau\alpha} - \mu_\tau(\rho_\tau, \partial_{\alpha_\tau}\rho_\tau)}{r_\tau(\rho_\tau)}}_{\text{local mean relaxation}} + \underbrace{\sigma_\tau(\rho_\tau)\ \eta_\alpha}_{\text{small-scale variability}}$$

Interpretation : j_τ relaxes to an average current μ_τ, at a rate r_τ, with variability of rms σ_τ.

$\Rightarrow$ Simple model, but we need to approximate these quantities from the mircodynamics.

$$\mathrm{d}j_{\tau\alpha} = \underbrace{-\frac{j_{\tau\alpha} - \mu_\tau(\rho_\tau, \partial_{\alpha_\tau}\rho_\tau)}{r_\tau(\rho_\tau)}}_{\text{local mean relaxation}} + \underbrace{\sigma_\tau(\rho_\tau)\ \eta_\alpha}_{\text{small-scale variability}}$$



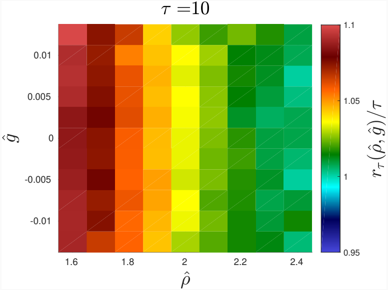


Avergage current : scale invariant (does not depend explicitly on τ) :

$$\mu_\tau(\rho_\tau, \partial_{\alpha_\tau}\rho_\tau) = \left(\frac{1}{4} - \frac{4}{q\rho_\tau^2}\right)\partial_{\alpha_\tau}\rho_\tau,$$

Sub-grid model : the relaxation rate

$$\mathrm{d}j_{\tau\alpha} = \underbrace{-\frac{j_{\tau\alpha} - \mu_\tau(\rho_\tau, \partial_{\alpha_\tau}\rho_\tau)}{r_\tau(\rho_\tau)}}_{\text{local mean relaxation}} + \underbrace{\sigma_\tau(\rho_\tau)\ \eta_\alpha}_{\text{small-scale variability}}$$

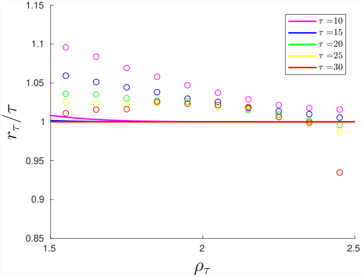


Relaxation rate : We have proposed a model based on equilibrium fluctuations :

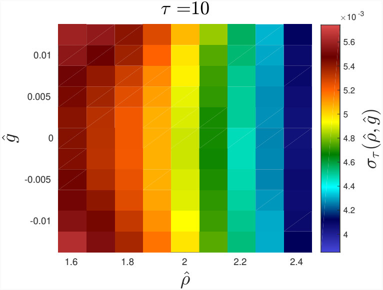
$$r_\tau(\rho_\tau) = \frac{1}{1 - \left(1 - \left(\frac{1}{\tau} + \frac{q}{8}\rho_\tau^2\right)\right)^\tau}$$

Depends on the dynamical parameter q.

$r_\tau \underset{\tau\to\infty}{\longrightarrow} 1$ corresponds to LTE.



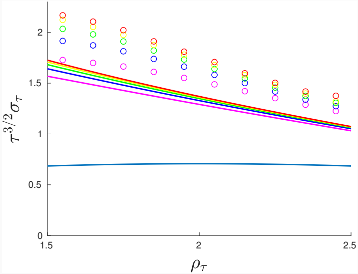
$$\mathrm{d}j_{\tau\alpha} = \underbrace{-\frac{j_{\tau\alpha} - \mu_\tau(\rho_\tau, \partial_{\alpha_\tau}\rho_\tau)}{r_\tau(\rho_\tau)}}_{\text{local mean relaxation}} + \underbrace{\sigma_\tau(\rho_\tau)\ \eta_\alpha}_{\text{small-scale variability}}$$



Gaussian fluctuations : Our equilibrium model gives the noise's rms :

$$\sigma_\tau(\rho_\tau) = \sqrt{\frac{\rho_\tau\left(1 - \frac{\rho_\tau}{4}\right)}{\tau^3\left(\frac{1}{\tau} + \frac{q}{8}\rho_\tau^2\right)}}$$

Depends on the dynamical parameter q.



$\sigma_\tau \underset{\tau\to\infty}{\to} 0$ corresponds to LTE (no fluctuations).
Different from the microscopic fluctuations $\sigma_*(\rho) = \sqrt{\frac{\rho}{2}\left(1 - \frac{\rho}{4}\right)}$.

Conclusion

Not always a separation of scale between micro-dynamics and macro-dynamics (observation) : mesoscopic modelling, geophysical and industrial turbulence, ...
$\Rightarrow$ Lag of the current : the relaxation to the "local equilibrium" is not complete for $\tau \ll \infty$.

Our model suggests that the coarse-grained current has to be considered as a dynamical variable, and that a stochastic relaxation is pertinent.

$$\partial_{t_\tau} \rho_\tau + \partial_{\alpha_\tau} j_{\tau\alpha} = 0,$$
$$\partial_{t_\tau} j_{\tau\alpha} = -\frac{j_{\tau\alpha} - \mu_\tau(\rho_\tau, \partial_{\alpha_\tau}, q)}{r_\tau(\rho_\tau, q)} + \sigma_\tau(\rho_\tau, q)\ \eta_\alpha.$$

This property is implicit in the famous $k - \epsilon$ model, but without fluctuations and no dependance of the empirical parameters on τ.

It would be interesting to find a refined sub-grid model using for example the Macroscopic Fluctuation Theory [Derrida, 2011, Bertini et al., 2014].

Perspectives

Perspectives

Data analysis : Apply this analysis to real flows : Rayleigh-Bénard convection, Poiseuille, Couette, grid/nozzle turbulence, Van-Karman, ...
... Any DNS or well resolved experimental data to analyse.

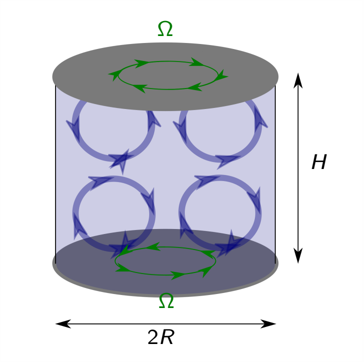
Theoretical :

1. One coarse-grained state $\rightarrow$ large (or even infinite) microscopic states. (Principle of Equilibrium SM)
2. One coarse grained state at a given time corresponds $\rightarrow$ large number (or even infinite) microscopic states that usually don't evolve to the same coarse-grained state.

$\Rightarrow$ Formalize it with the Nonequilibrium Statistical Physics/Dynamical system formalism to find the right stochastic model, with a particular care on the coarse-graining procedure. $\sim$ What is μ_τ, r_τ, and $\sigma_\tau \eta$?

Work in progress

Van-Karman flow :



Incompressible Navier-Stokes :

$$\partial_\alpha u_\alpha = 0,$$
$$\partial_t u_\alpha + u_\beta \partial_\beta u_\alpha = -\partial_\alpha p + g_\alpha + \partial_\beta^2 u_\alpha.$$

$\Rightarrow$ Vorticity equation :

$$\omega_\alpha = (\nabla \times u)_\alpha,$$
$$\partial_t \omega_\alpha = (\nabla \times F)_\alpha,$$
$$F_\alpha = (u \times \omega)_\alpha - \mu(\nabla \times \omega)_\alpha.$$

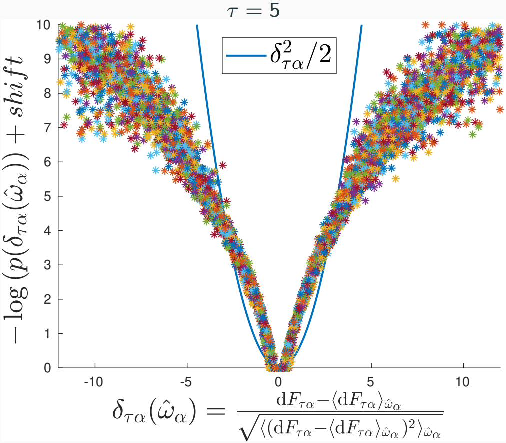
Thanks to Paul Debue, Adam Cheminet, and others members of the SPHYNX for the well resolved experimental velocity fields (at the center of the cell).

Coarse-grained dynamics ? We propose

$$\omega_{\tau\alpha} = (\nabla_\tau \times u_\tau)_\alpha,$$
$$\partial_{t_\tau} \omega_{\tau\alpha} = (\nabla_\tau \times F_\tau)_\alpha,$$
$$F_{\tau\alpha} = (u_\tau \times \omega_\tau)_\alpha - \mu(\nabla_\tau \times \omega_\tau)_\alpha.$$

What is $p(\mathrm{d}F_{\tau\alpha}|\omega_{\tau\alpha})$?

Results for $\alpha = x, y, z$ and various τ :



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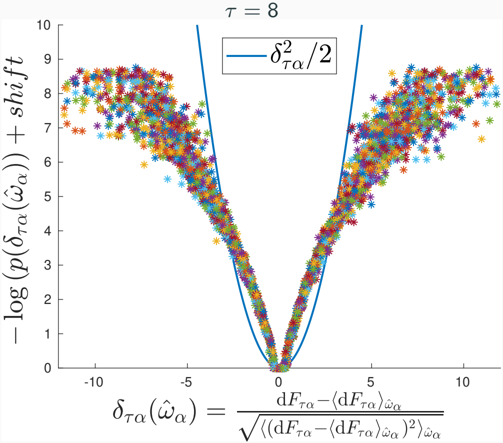

Work in progress

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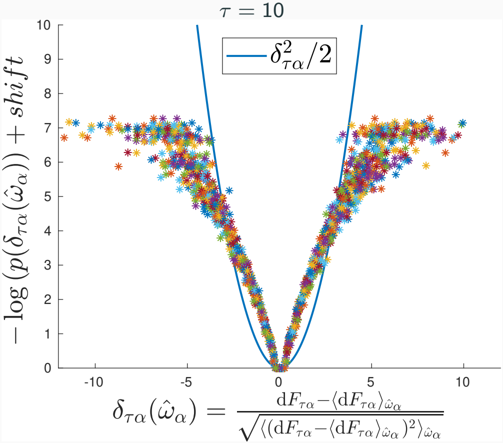

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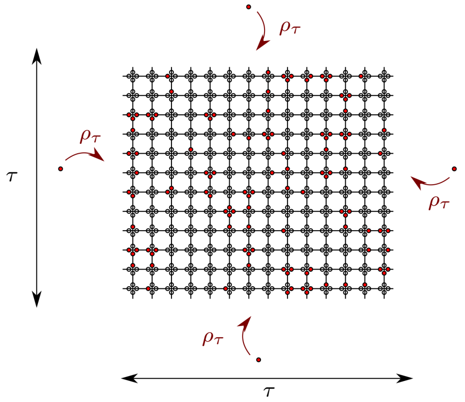
Sub-grid model

Simplifications :

- Instantaneous mixing of particles at each discrete time step $\rightarrow$ dynamics of the spatial average occupations $N_{si}(t_*)$, $i = 1, ..., 4$;
- Variations at each time step modelled as a combination of random variables that take into account entry/outcome at boundaries and collisions in the bulk.



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Example : for N_{s1}

Source/Sink	nodes	probability distribution
entry (left boundary)	τ	$\begin{cases} P(X=-1)=0, \\ P(X=0)=1-\rho_\tau/4, \\ P(X=1)=\rho_\tau/4. \end{cases}$
exit (right boundary)	τ	$\begin{cases} P(X=-1)=N_{s1}, \\ P(X=0)=1-N_{s1}, \\ P(X=1)=0. \end{cases}$
2 part collision	τ^2	$\begin{cases} P(X=-1)=N_{s1}(1-N_{s2})N_{s3}(1-N_{s4})p, \\ P(X=0)=1-P(X=-1)-P(X=1), \\ P(X=1)=(1-N_{s1})N_{s2}(1-N_{s3})N_{s4}p. \end{cases}$
3 part collision	τ^2	$\begin{cases} P(X=-1)=N_{s1}N_{s2}(1-N_{s3})N_{s4}q, \\ P(X=0)=1-P(X=-1)-P(X=1), \\ P(X=1)=(1-N_{s1})N_{s2}N_{s3}N_{s4}q. \end{cases}$

Sub-grid model

τ sufficiently large for using the central limit theorem and we linearise around the mean occupation of channels $\rho_\tau/4$: $N_{si}(t_*) \simeq \rho_\tau/4 + \delta N_{si}(t_*), i = 1, ..., 4$ with $\delta N_{si}(t_*) << 1$.

$$j_{sx}(t_* + 1) - j_{sx}(t_*) = -\left(\frac{1}{\tau} + \frac{q}{8}\rho_\tau^2\right) j_{sx} + \sqrt{\frac{1}{\tau^2}\rho_\tau \left(1 - \frac{\rho_\tau}{4}\right)\left(\frac{1}{\tau} + \frac{q}{8}\rho_\tau^2\right)}\, \eta_x$$

where η_x is a normal random variable. It is a linear Langevin equation that arise in many contexts in Physics [Gardiner, 2009]. The pdf of the coarse-grained (space and time averaged current)

$$j_{\tau x}(t_\tau) = \frac{1}{\tau} \sum_{t_*=\tau t_\tau}^{\tau(t_\tau+1)-1} j_{sx}(t_*),$$

satisfies a large deviation principle [Touchette, 2018] :

$$\lim_{\tau\to\infty} -\frac{1}{\tau} \ln P(j_{\tau x} = j) = I(j)$$

$$I(j) = \frac{1}{2} \frac{(\frac{1}{\tau} + \frac{q}{8}\rho_\tau^2)^2}{\frac{1}{\tau^2}\rho_\tau(1 - \frac{\rho_\tau}{4})(\frac{1}{\tau} + \frac{q}{8}\rho_\tau^2)} j^2 = \frac{\tau^2}{2} \frac{(\frac{1}{\tau} + \frac{q}{8}\rho_\tau^2)}{\rho_\tau(1 - \frac{\rho_\tau}{4})} j^2.$$

Therefore, our model states that $j_{\tau x}$ is a Gaussian random variable with mean zero and rms

$$\sigma_\tau(\rho_\tau, q) = \sqrt{\frac{\rho_\tau(1 - \frac{\rho_\tau}{4})}{\tau^3(\frac{1}{\tau} + \frac{q}{8}\rho_\tau^2)}}.$$

The evolution of $\langle j_{sx} \rangle$ in the mesocell can be written

$$\langle j_{sx} \rangle(t_* + 1) = a \langle j_{sx} \rangle(t_*), \qquad a = 1 - \left(\frac{1}{\tau} + \frac{q}{8}\rho_\tau^2\right).$$

It is easy to show that

$$\langle j_{\tau x} \rangle(t_\tau + 1) - \langle j_{\tau x} \rangle(t_\tau) = -(1 - a^\tau)\langle j_{\tau x} \rangle(t_\tau), \qquad a \equiv 1 - \left(\frac{1}{\tau} + \frac{q}{8}\rho_\tau^2\right)$$

This equation suggests that the relaxation time for the coarse-grained current is

$$r_\tau(\rho_\tau, q) = \frac{1}{1 - a^\tau} = \frac{1}{1 - \left(1 - \left(\frac{1}{\tau} + \frac{q}{8}\rho_\tau^2\right)\right)^\tau}.$$

Since $-1 - \frac{1}{\tau} = a(\rho_\tau = 4) \leq a \leq (\rho_\tau = 0) = 1 - \frac{1}{\tau}$, in normal conditions (ρ_τ not too close from 4) $r_\tau \underset{\tau \to \infty}{\longrightarrow} 1$.