Anomalous diffusion and intermittency in random dynamical systems

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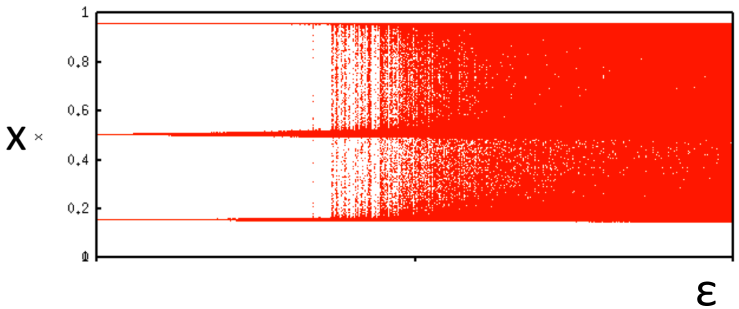
Random dynamical systems theory for nonlinear stochastic phenomena

Random logistic map

$$x_{n+1} = a x_n (1 - x_n) + \xi_n$$

$$a = 3.83$$

$$\xi_n : \text{ bounded uniform noise in } [\epsilon, -\epsilon]$$



[G. Mayer-Kress and H. Haken, 1981
YS, T-S Doan, M, Rasmussen, J. Lamb, submitting]

Stochastic Lorenz equation

$$\begin{cases} dx = s(y-x)dt + \sigma x \, dW_t, \\ dy = (rx - y - xz)dt + \sigma y \, dW_t, \\ dz = (-bz + xy)dt + \sigma z \, dW_t. \end{cases}$$

r = 28, s = 10, b = 8/3 , σ = 0.3
Wt: Wiener process

[M. Chekroun, E. Simonnet, M. Ghil, 2011
YS, M. Chekroun, M. Ghil, in preparation]

Outline

1. Random dynamical systems and noise-induced phenonema

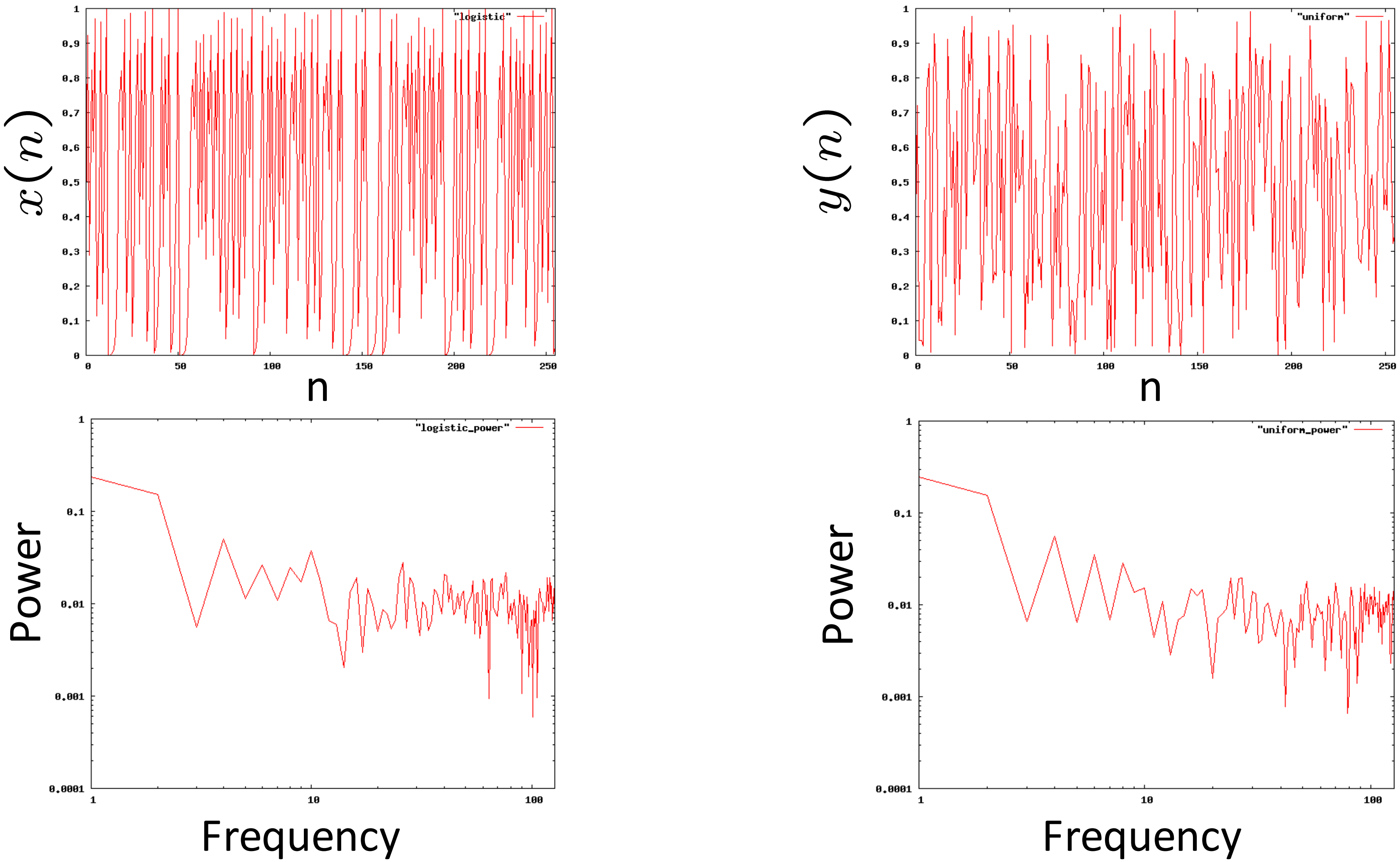
2. Deterministic diffusion

3. Anomalous diffusion in random dynamical systems

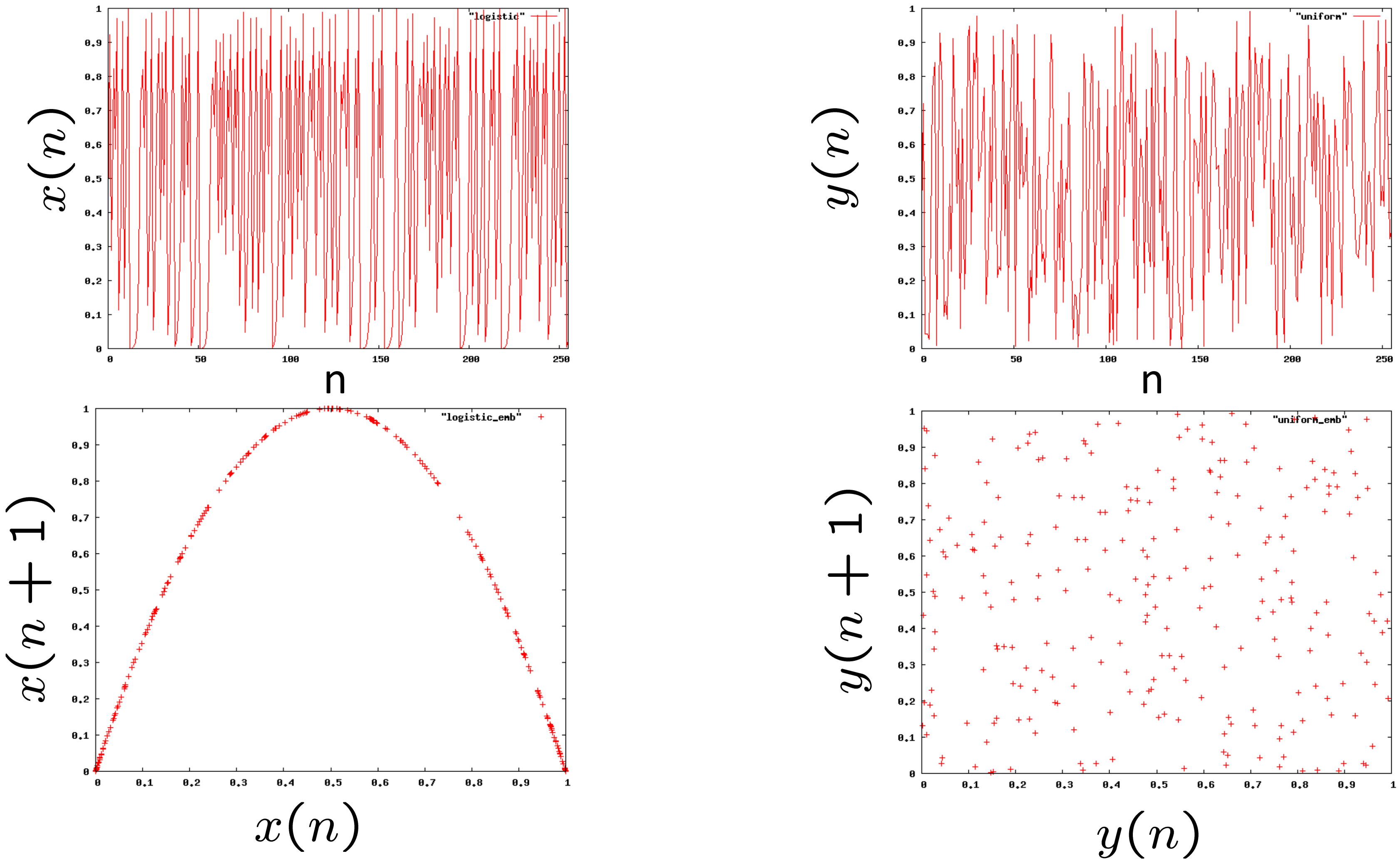
4. Summary and future projects

1. Random dynamical systems and noise-induced phenomena

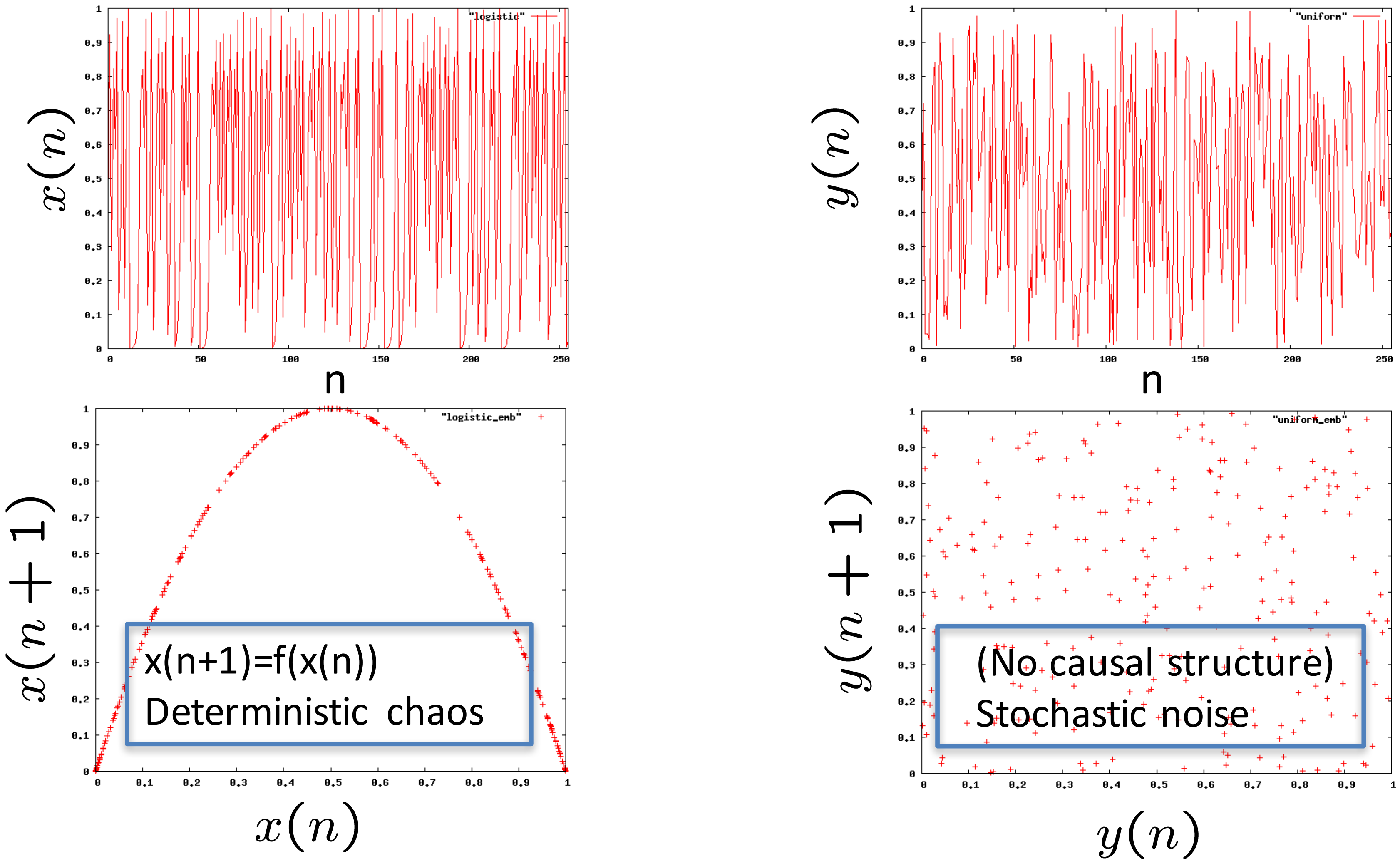
Irregular time series



First return plot

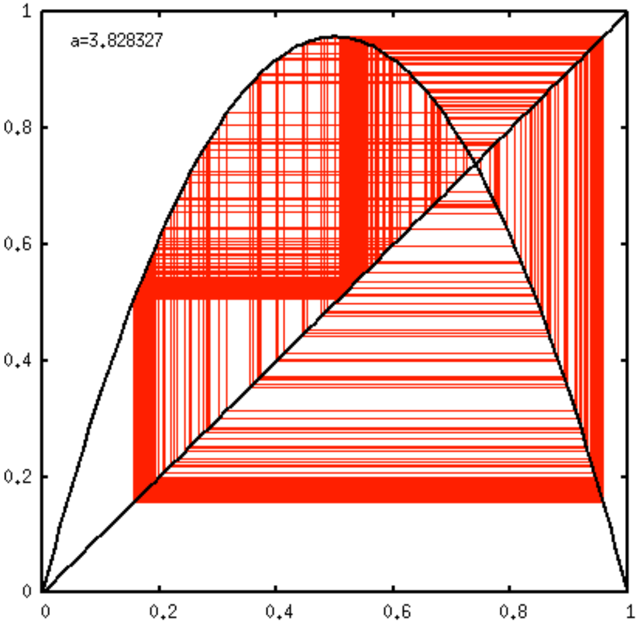


Extracting dynamics from data



Extracting dynamics from data

x_{n+1}

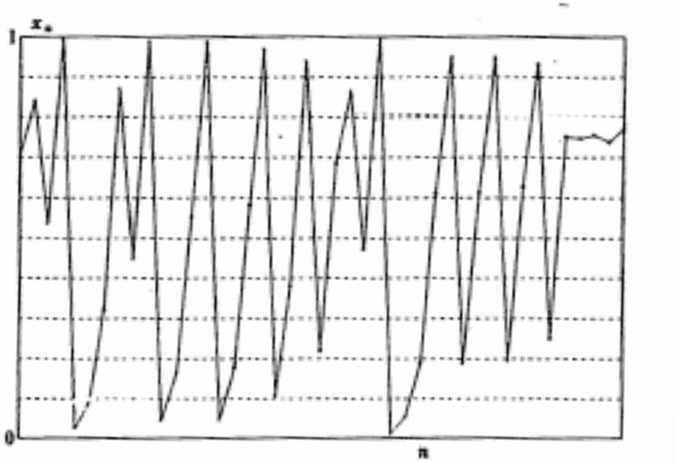


x_n

Logistic map [May, 1976]

$$x_{n+1} = ax_n(1 - x_n)$$

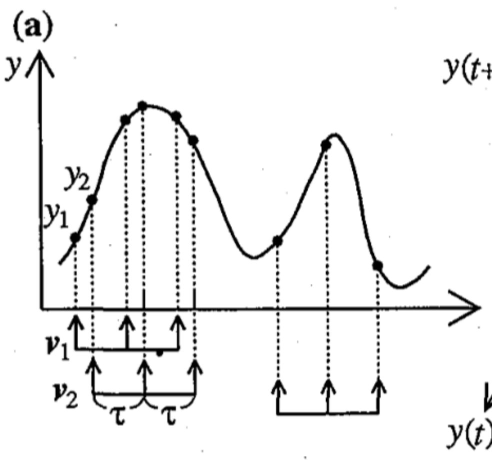
x_n

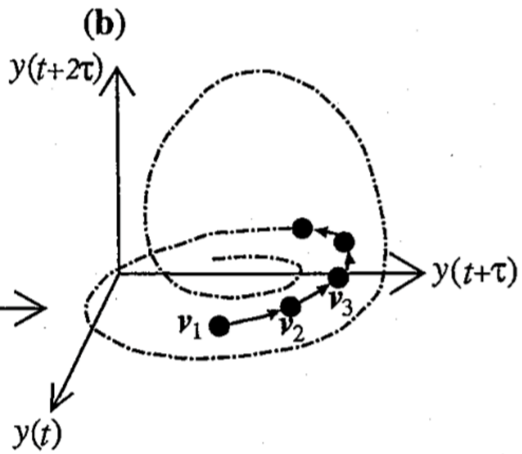


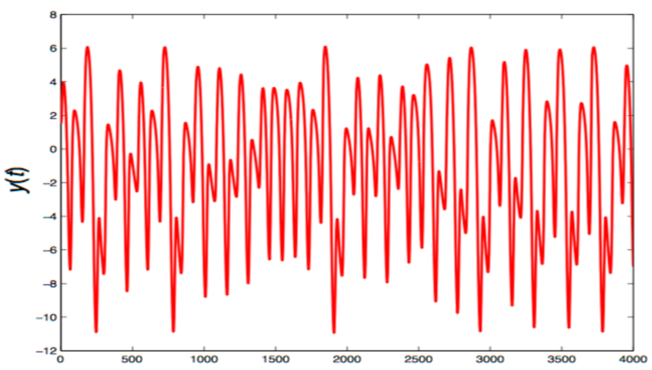
n

Delay coordinate plot and embedding

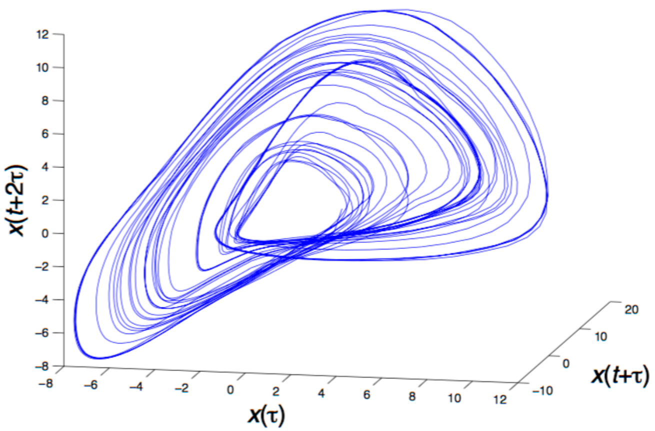
- Attractor reconstruction





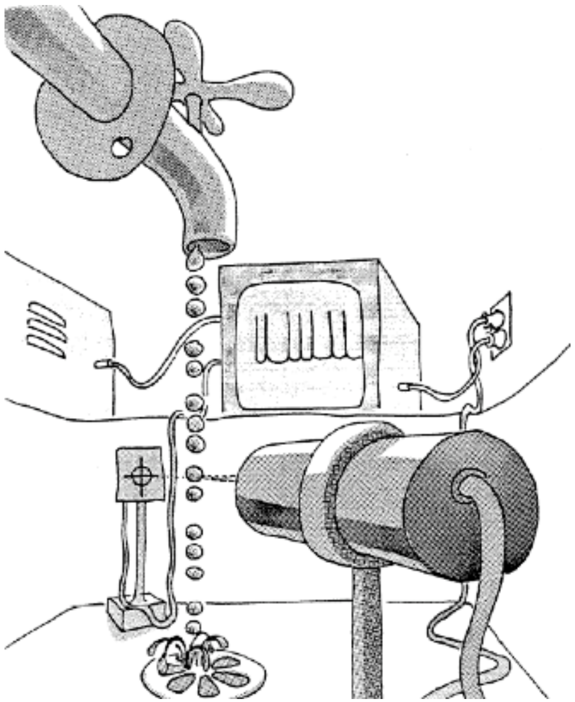


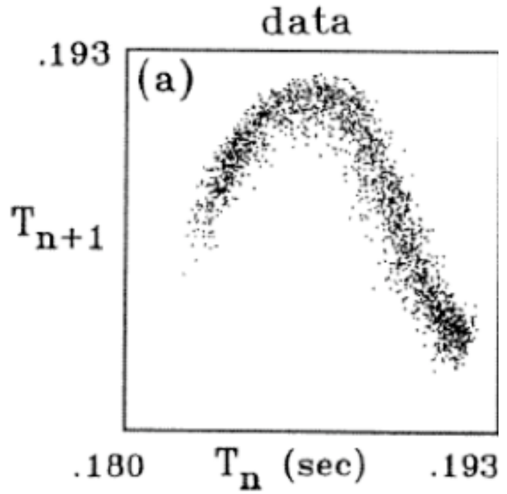
Rossler attractor



Return plot of experimental data

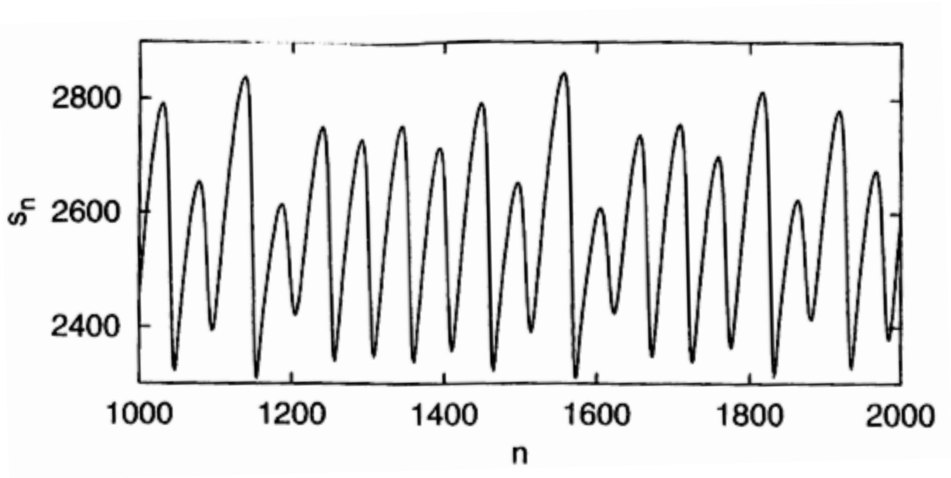
Chaos in dripping faucet [Shaw, et. al. 1984]

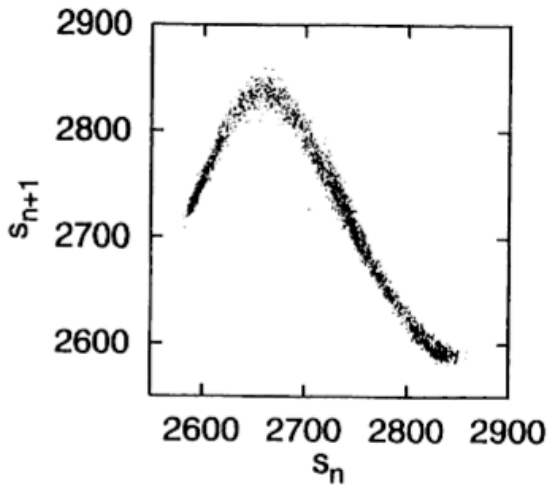




Return plot of experimental data

Nonlienar laser with feedback [Arrecci, et. al., 1986]





Return plot for experimental data

Belousov-Zhabotinskii chemical reaction [R. H. Simoyi, et. al., 1982]

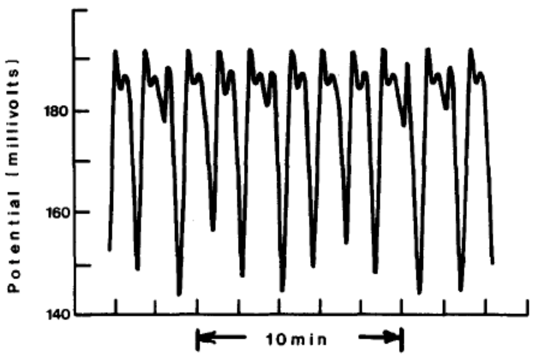


FIG. 4. Recording from bromide ion electrode; T = 25 °C; flow rate = 4.31 ml/min; Ce^{+3} catalyst.

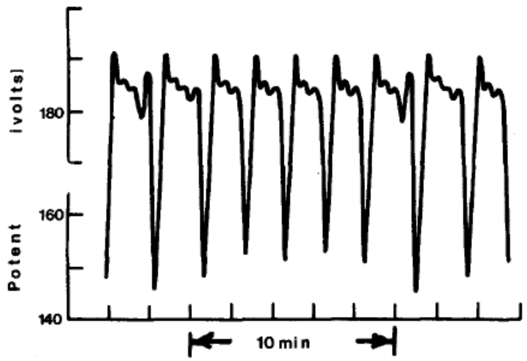
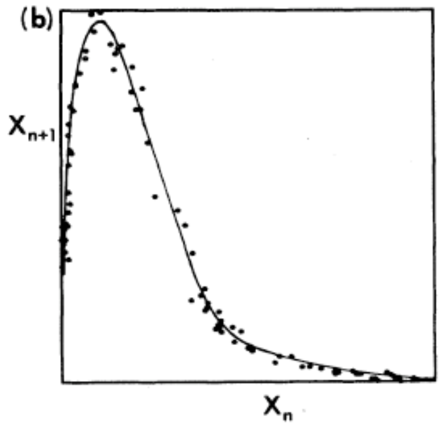
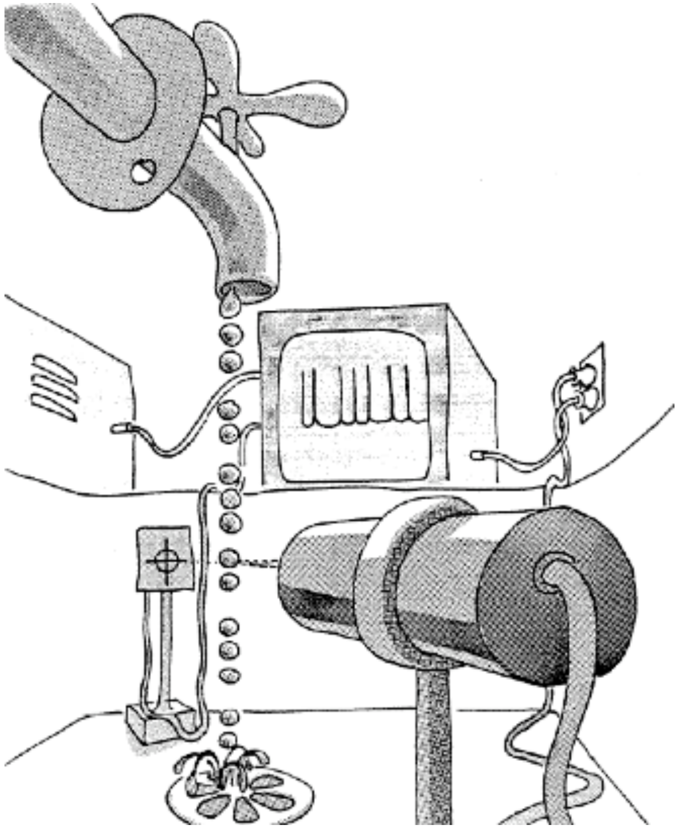


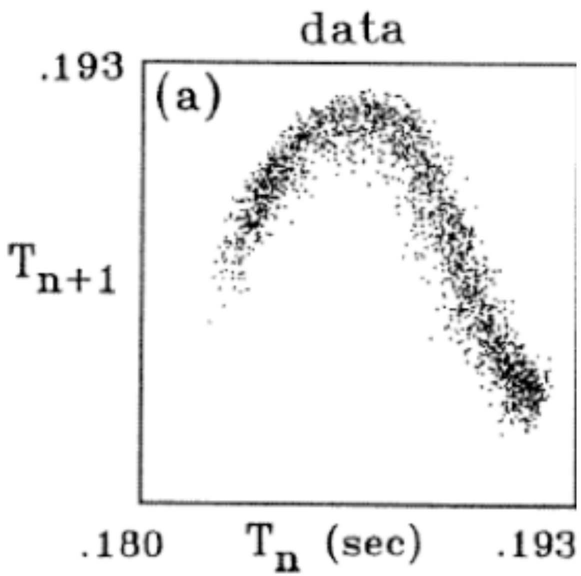
FIG. 6. Recording from bromide ion electrode; T = 25 °C; flow rate = 4.51 ml/min; Ce^{+3} catalyst.



Chaotic dynamics

Chaos in dripping faucet

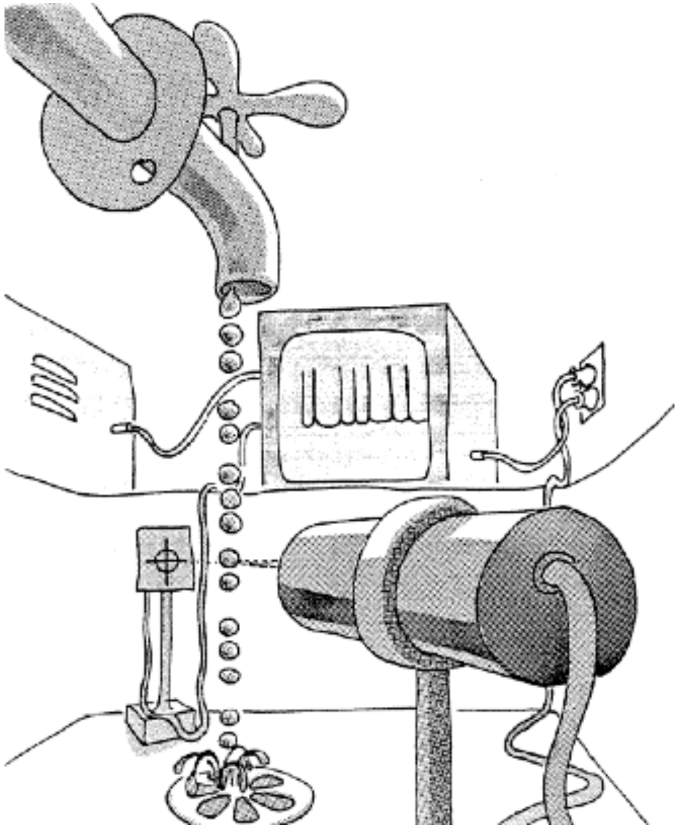


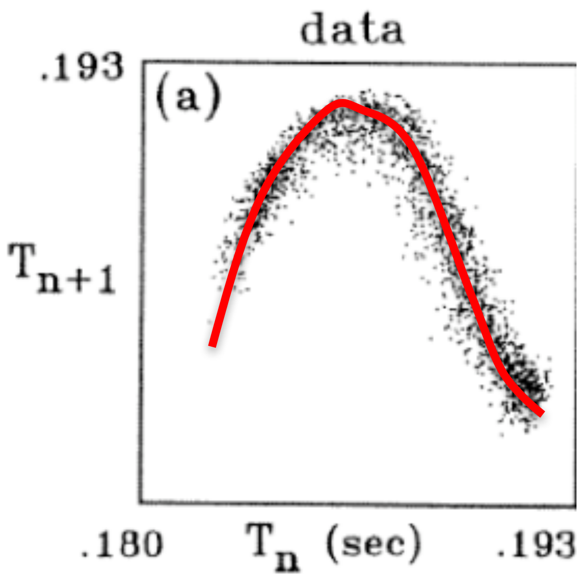


[R. Shaw, 1984]

Dynamical system model

Chaos in dripping faucet

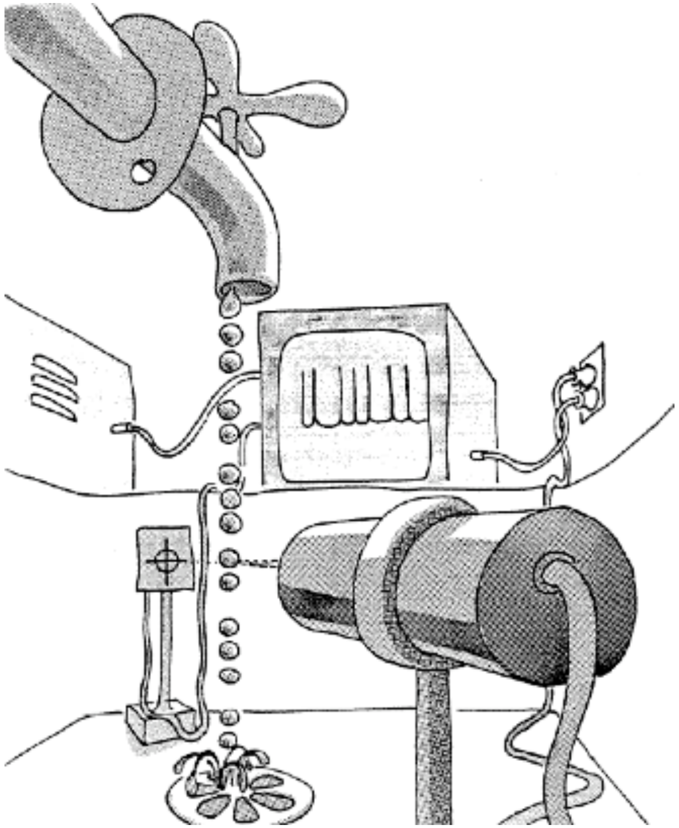


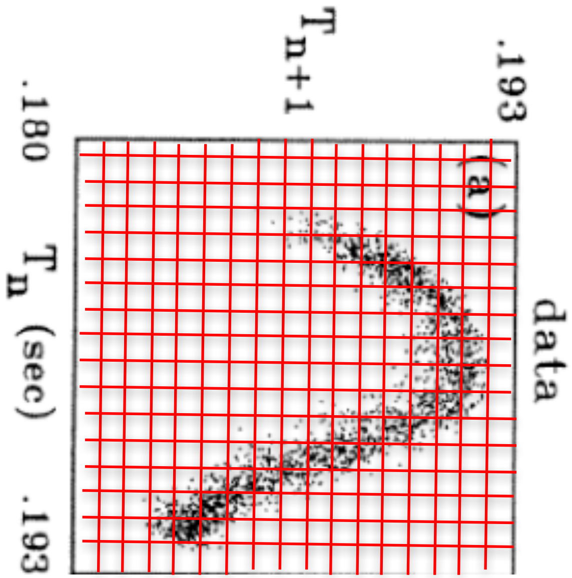


$$x_{n+1} = f(x_n)$$

Stochastic process

Chaos in dripping faucet

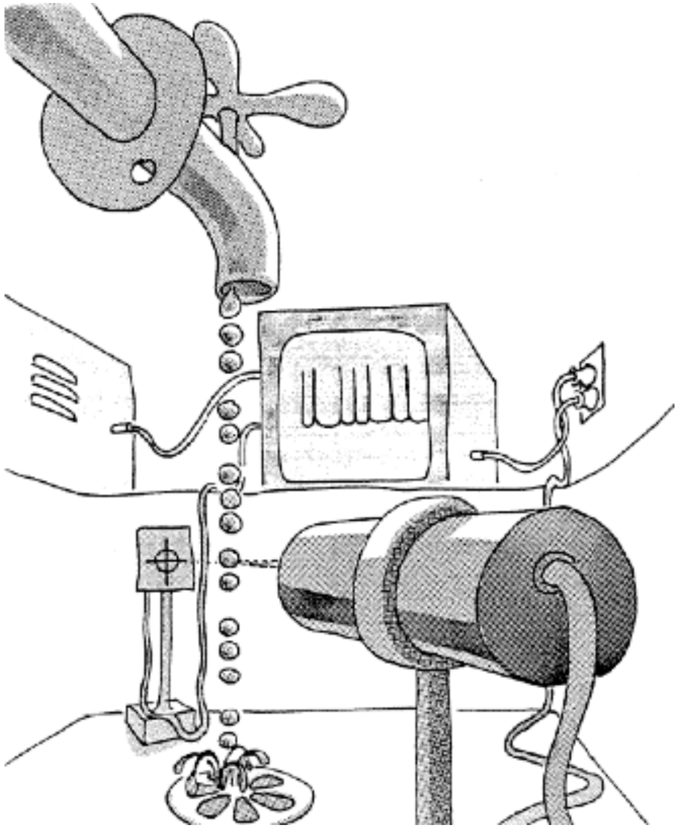


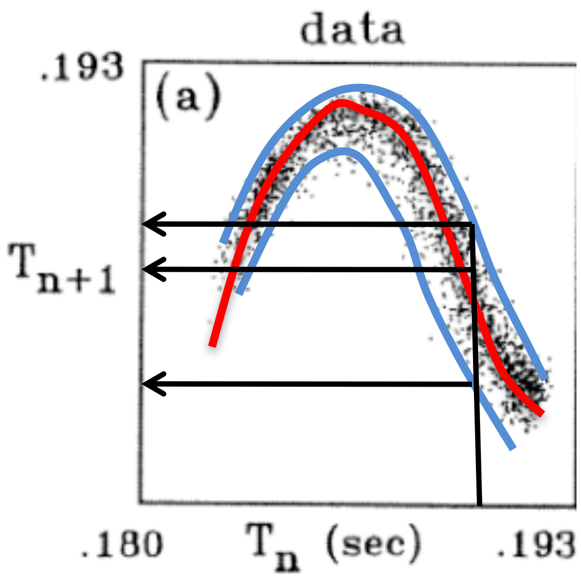


$$\rho_{n+1} = P\rho_n$$

Random dynamical system model

Chaos in dripping faucet

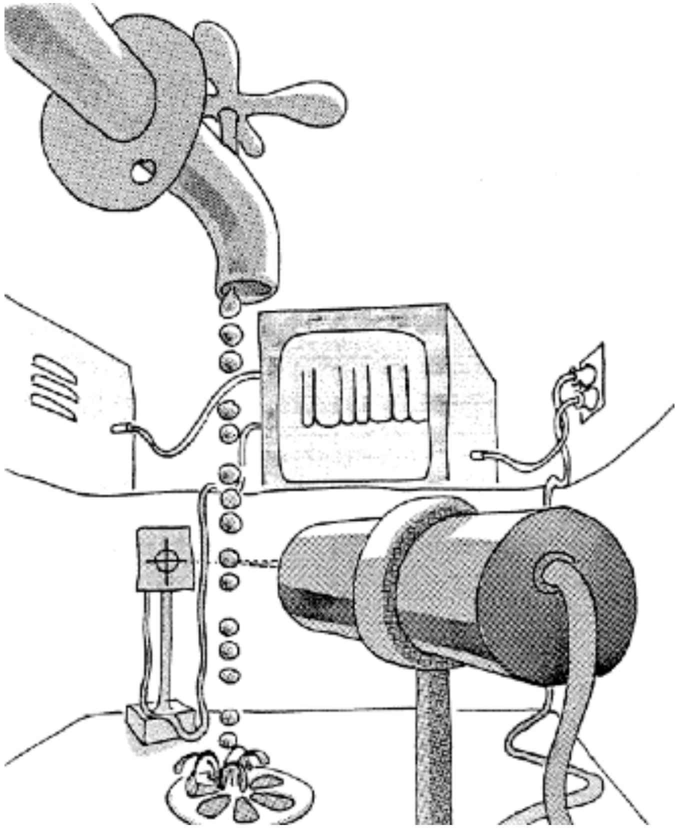


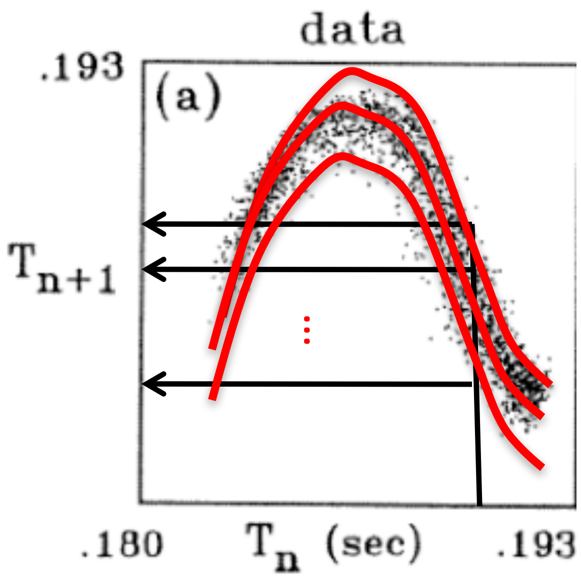


$$x_{n+1} = f(x_n, \xi_n)$$

Non-autonomous dynamical system model

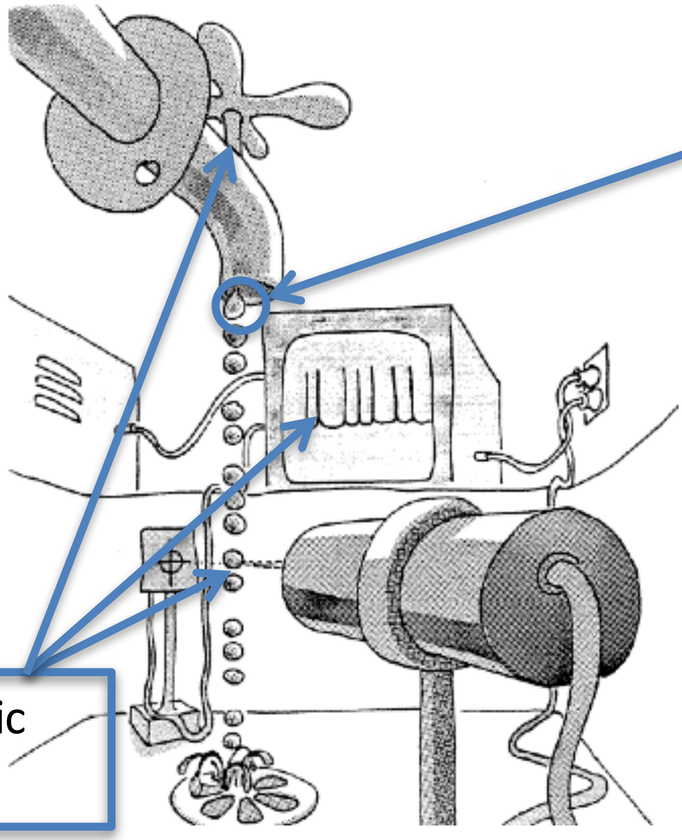
Chaos in dripping faucet



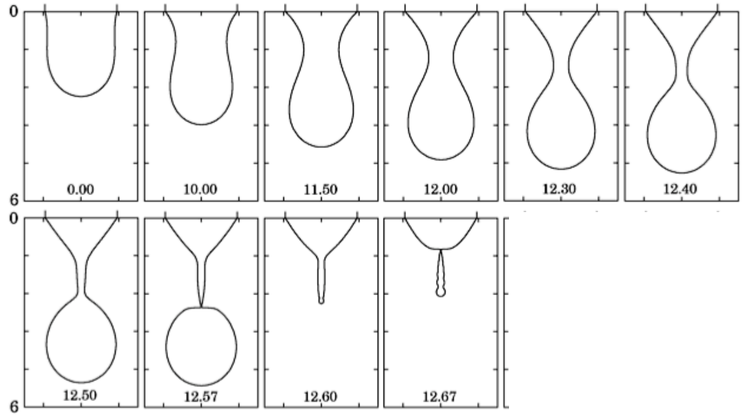


$$x_{n+1} = f_n(x_n)$$

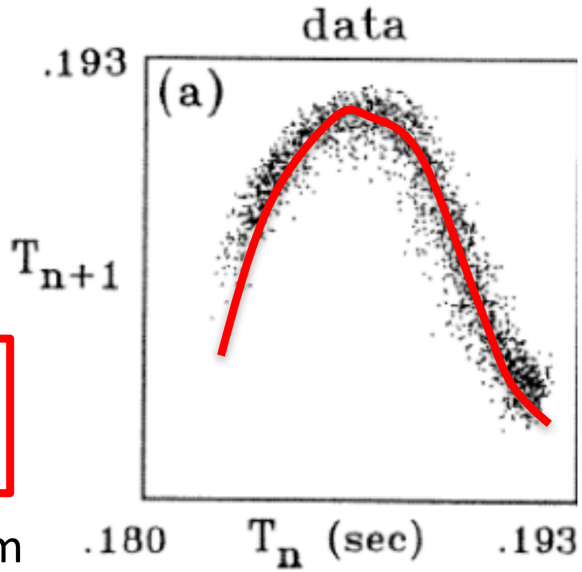
Dynamical systems with a large degrees of freedom



Extrinsic noise



Intrinsic noise



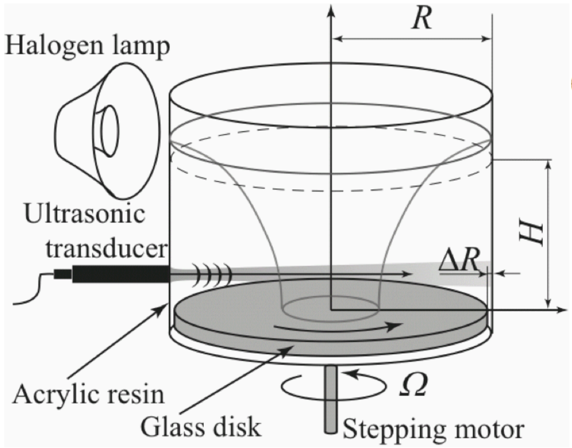
$$x_{n+1} = f(x_n, \xi_n) \text{ or } x_{n+1} = f_n(x_n)$$

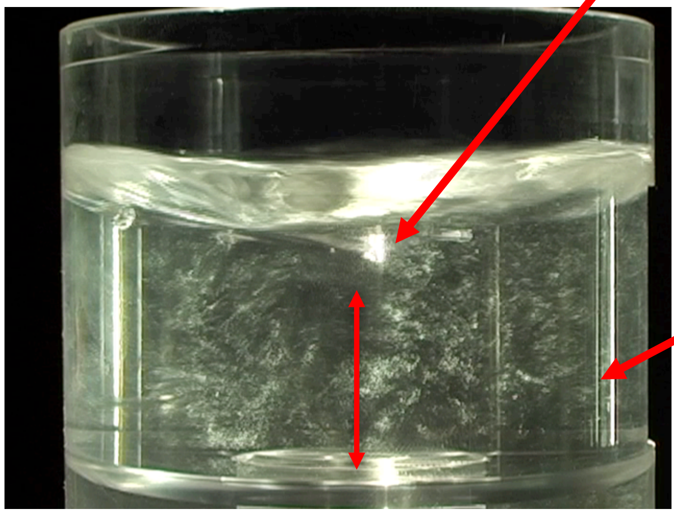
Stochastic term

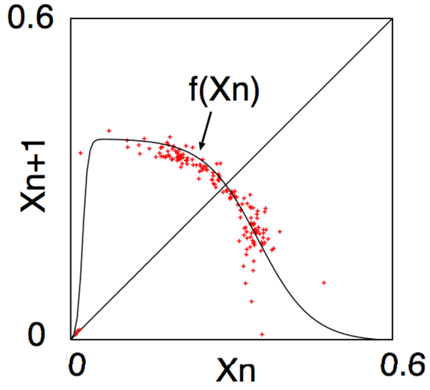
Non-autonomous term

Random dynamics from time series of rotating fluid

[M. Iima, et.al, (2006)]

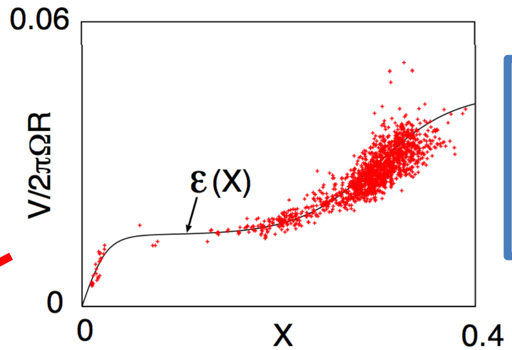






Slow motion of
surface height
〜 1D dynamics

$$f(x) = A\left[\frac{1}{1+e^{-\beta_1(x-p)}} - \frac{1}{1+e^{\beta_1 p}} + \frac{1}{1+e^{-\beta_2(x-q)}} - \frac{1}{1+e^{\beta_2 q}}\right]$$



Fast motion
of rotating fluid
〜 Noise

$$\epsilon(x) = B\left[\frac{e^{\beta_3 r} - e^{-\beta_3(x-r)}}{(1+e^{-\beta_3(x-r)})(1+e^{\beta_3 r})} + \frac{1-e^{-\beta_4 x}}{2(1+e^{-\beta_4 x})} + \delta\right]$$

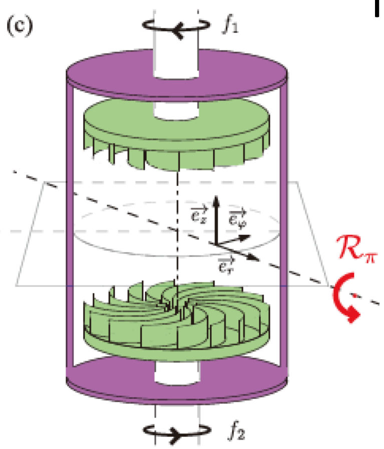
random return map $x_{n+1} = f(x_n) + \epsilon(x_n)\xi_n$

ξn: White Gaussian noise

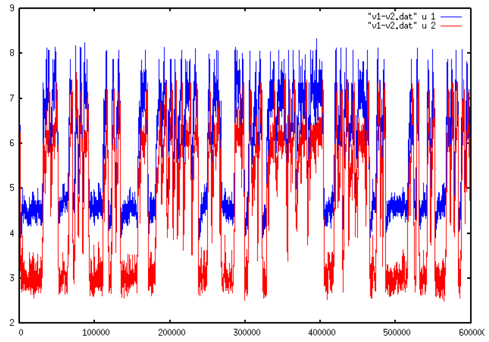
[YS, et. al., 2010]

Stochastic chaos in a turbulent swirling flow

Collective motion in Karman flow



f1.f2

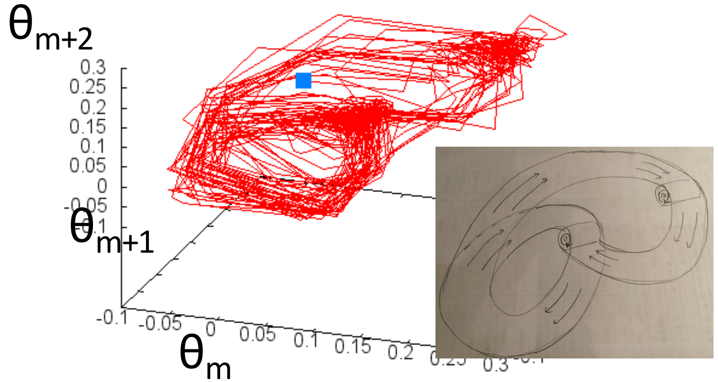


t

[B. Saint-Michel, et.al, 2013]

Time series embedding

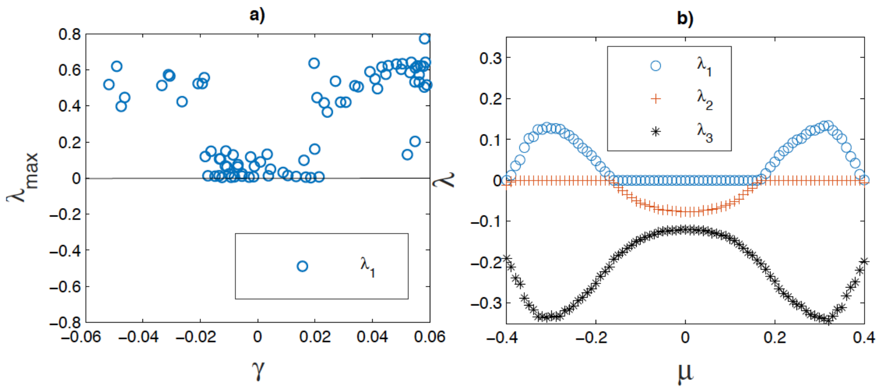
$$\theta = (f_1 - f_2)/(f_1 + f_2)$$



Stochastic Duffing equation

$$\boxed{\begin{aligned} dx &= ydt \\ dy &= (-ay + x - x^3 + z\sin(\omega t))dt \\ dz &= -\phi(z-\mu)dt + \sigma dW_t \end{aligned}}$$

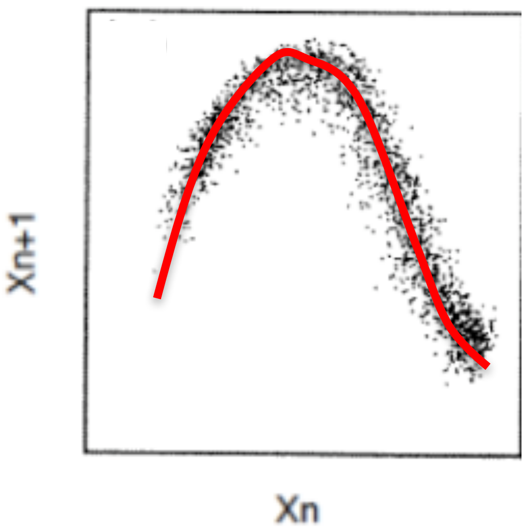
Lyapunov spectrum



[D. Faranda, YS, B. Saint-Michel, C. Wiertel, V. Padilla, B. Dubrulle, F. Daviaud., PRL, 2017]

One-dimensional maps with presence of noise

$$x_{n+1} = f(x_n) \quad + \quad \xi_n$$

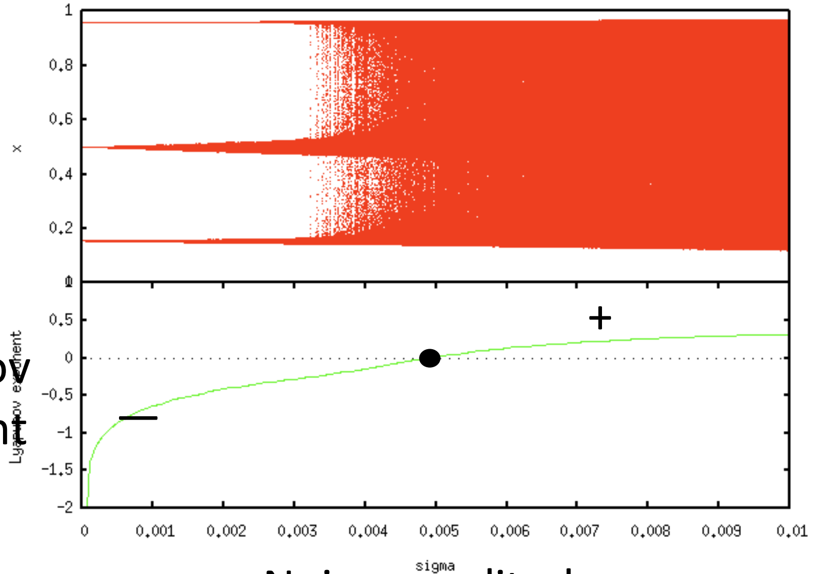


ξn: Noise

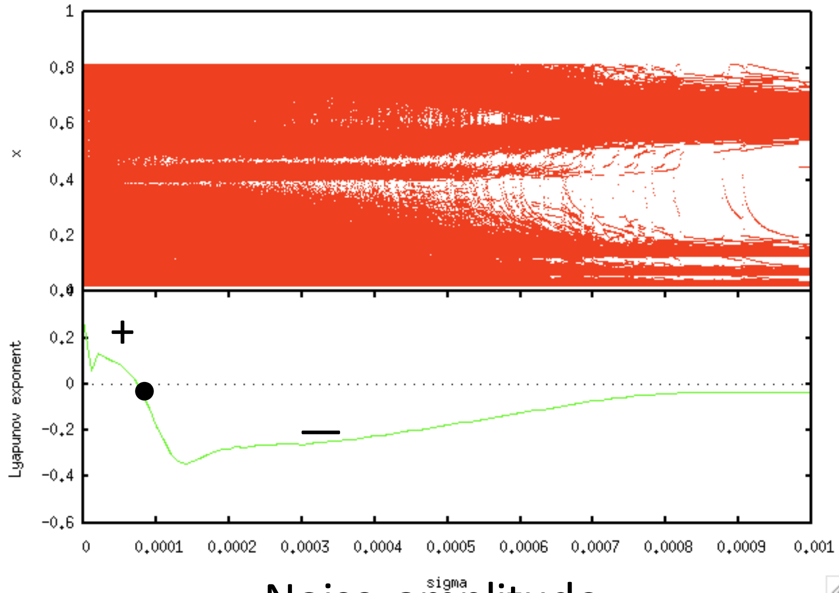
One-dimensional random maps

Orbit

Lyapunov exponent

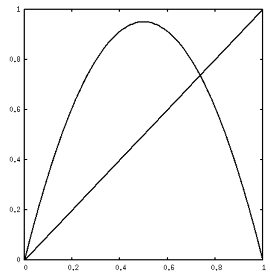


Noise amplitude



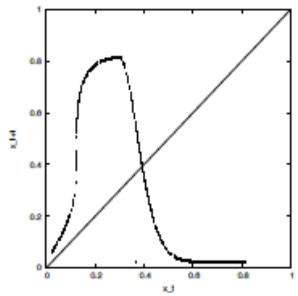
Noise amplitude

$$x_{n+1} = f(x_n) + \xi_n$$



Noise-induced chaos
in logistic map

$$x_{n+1} = f(x_n) + \xi_n$$



Noise-induced order
in BZ map

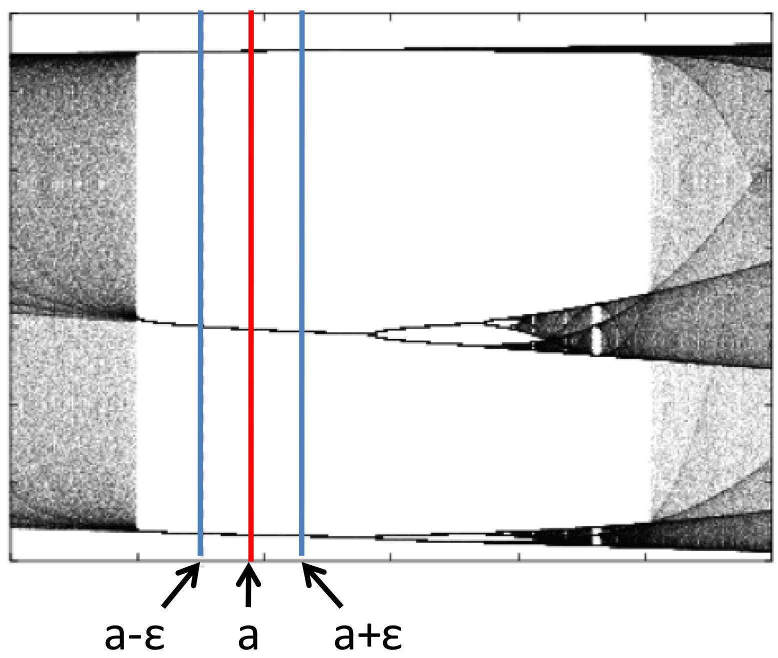
[YS, T-S Doan, M, Rasmussen,
J. Lamb, submitting]

[S. Galatolo, et. al.,
https://arxiv.org/abs/1702.07024]

Noise-induced chaos

"Is period 3 logistic map in window region potentially chaotic under noisy measurements?"

Model: $x_{n+1} = a - x_n^2 + \epsilon\xi_n$ (a=1.755, ξ∈[-1,1]: noise)



Noise-induced chaos

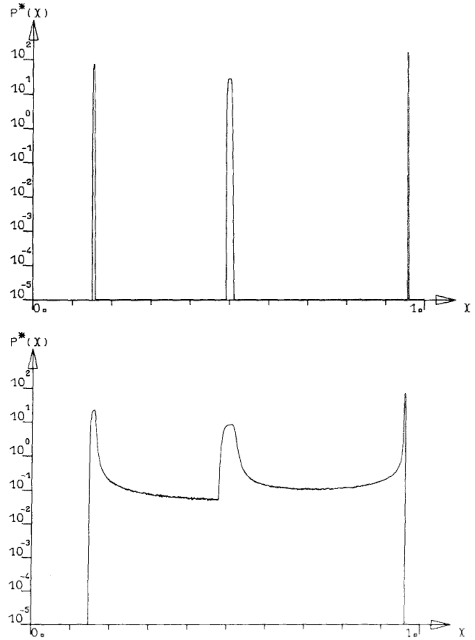
Small additive noise to period 3 window region makes non-attracting chaotic set observable.

$$x_{n+1} = a - x_n^2 + \epsilon\xi_n \quad \text{(a=1.755, } \xi \in \text{[-1,1]: noise)}$$

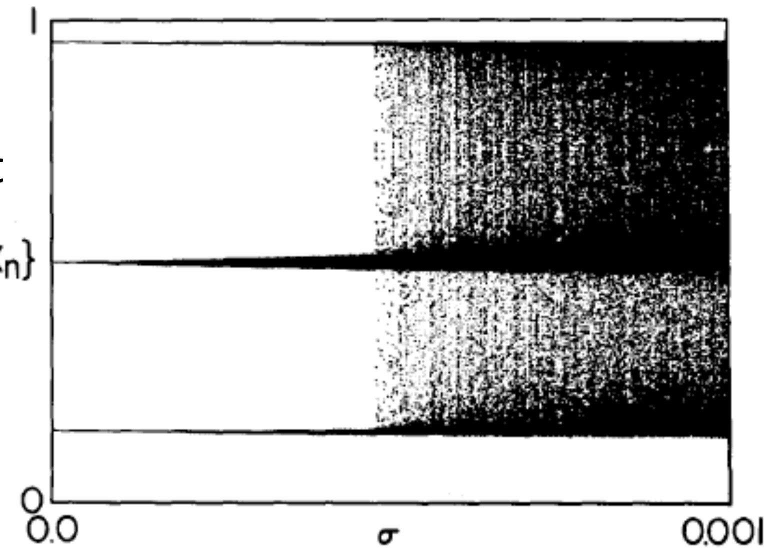
Without noise

Invariant densities

With noise



orbit $\{X_n\}$



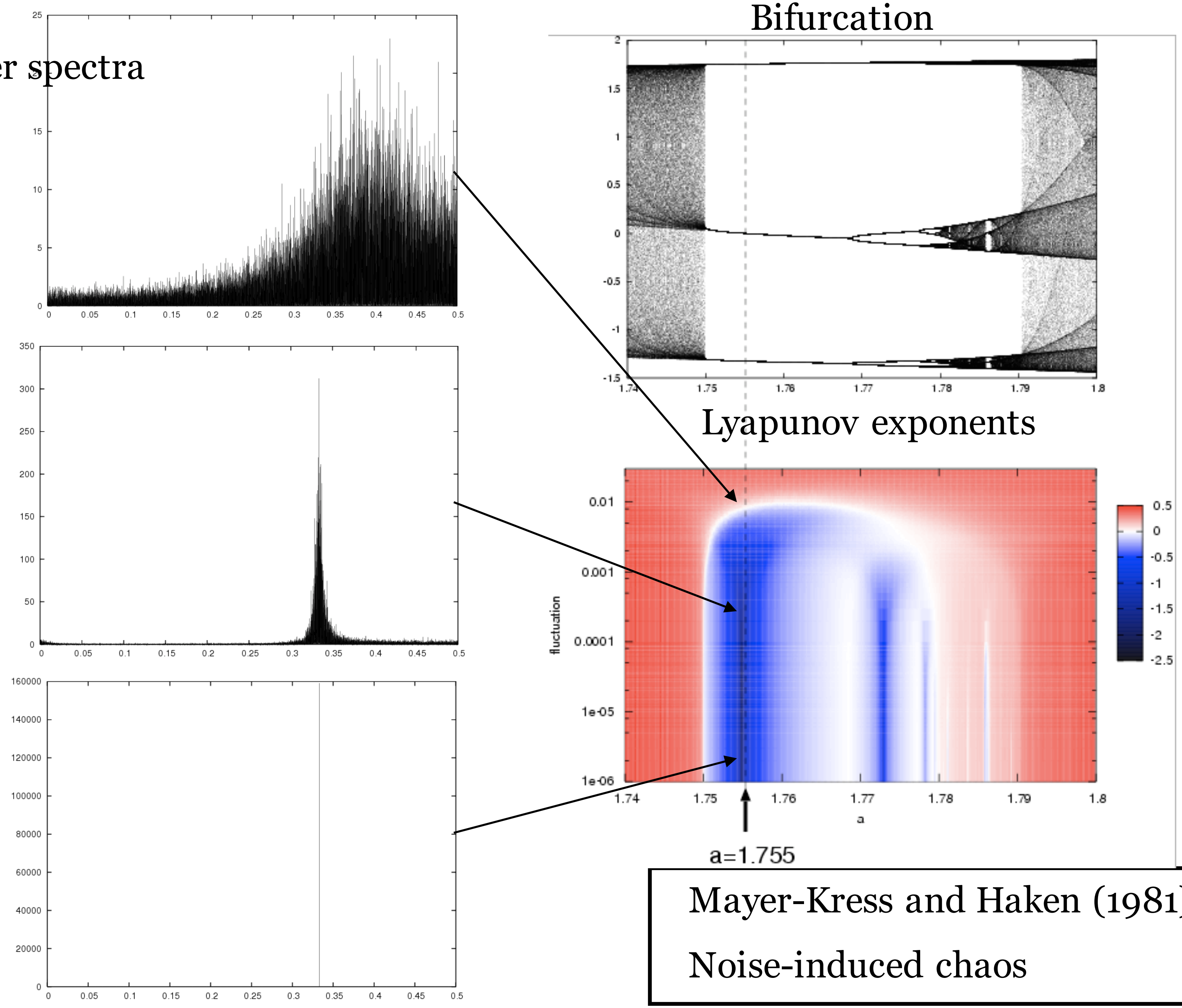
Fluctuation size

[Crutchfield et.al., 1982]

Power spectra

Bifurcation

Lyapunov exponents



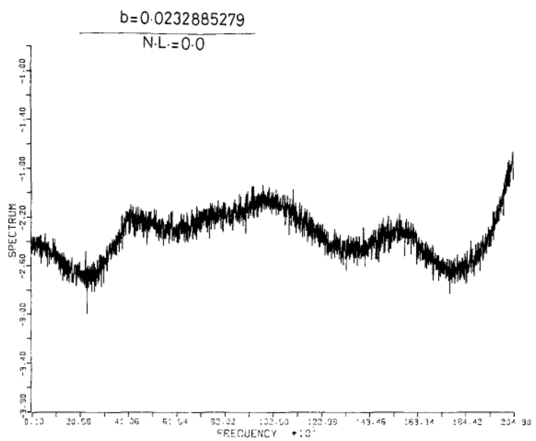
Mayer-Kress and Haken (1981)

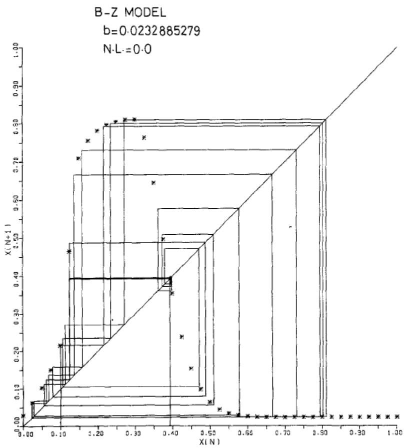
Noise-induced chaos

Noise-induced order

- Small additive noise to chaotic region of BZ maps induces a peak of power spectrum.

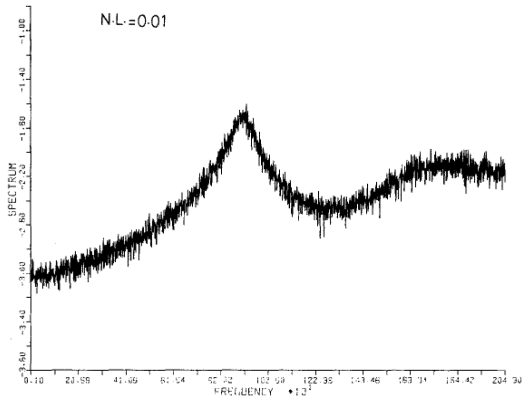
Without noise

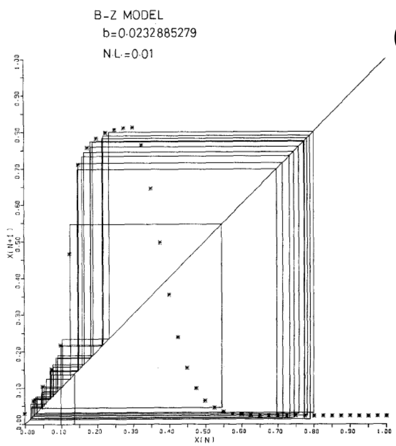




Power
Spectra

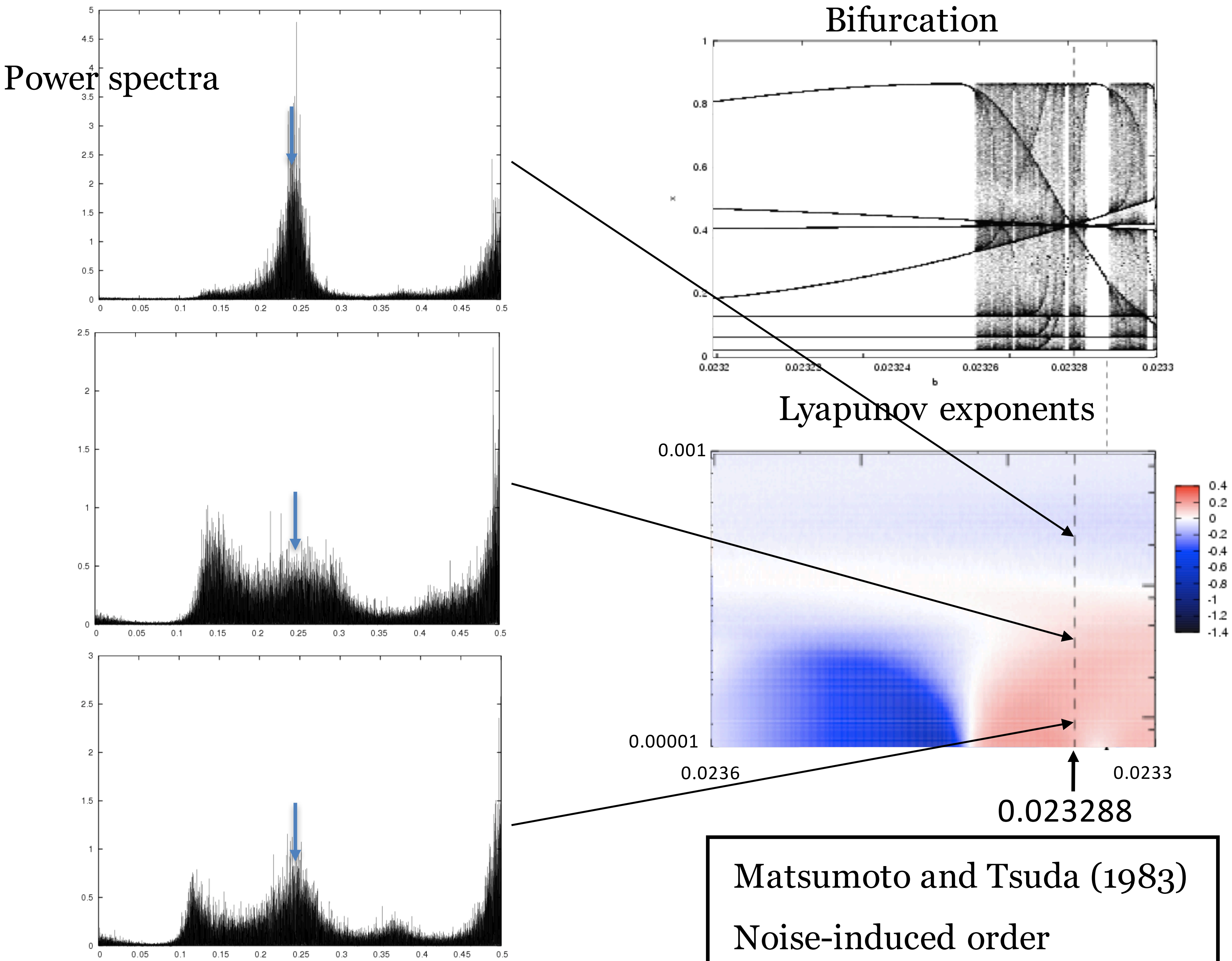
With noise





Orbits

[Matsumoto, Tsuda, 1983]

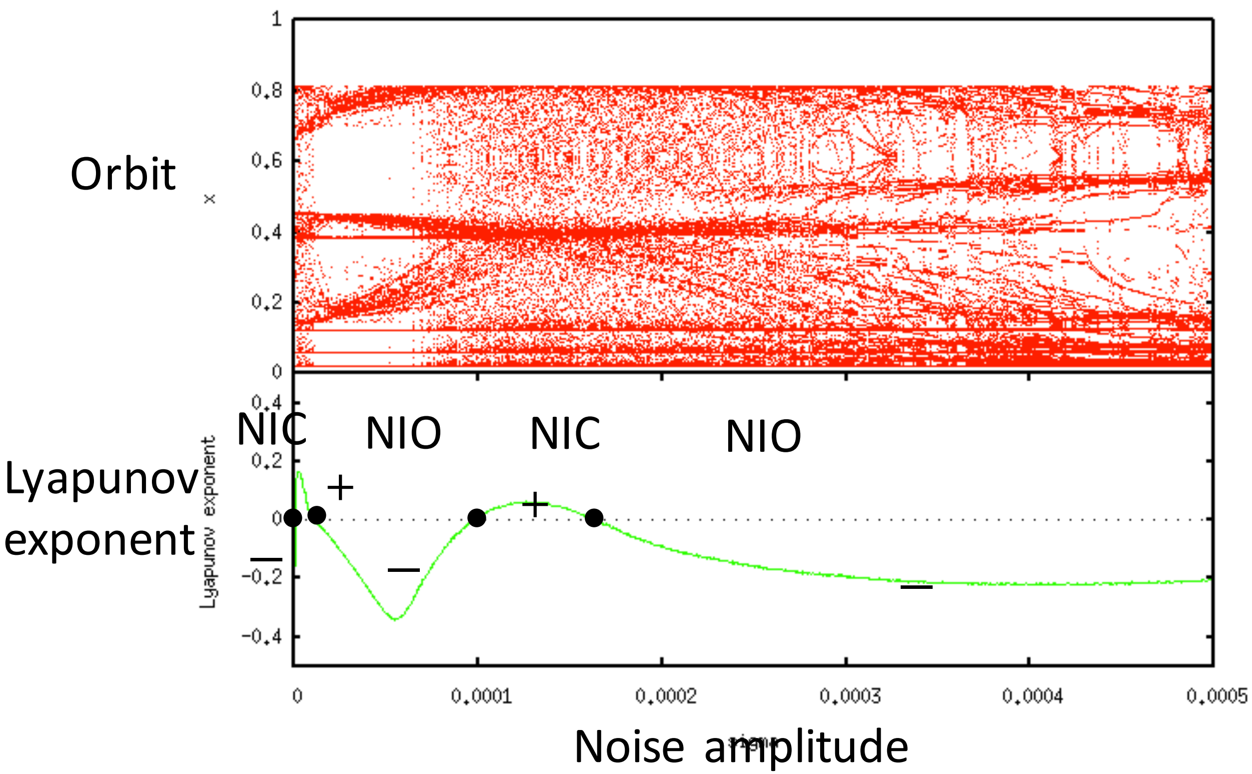


Matsumoto and Tsuda (1983)

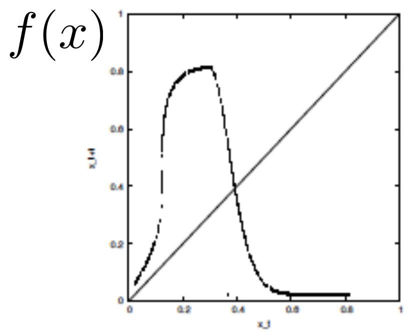
Noise-induced order

Multiple noise-induced transition

Both Noise-induced chaos (NIC) and noise-induced order (NIO) are observed increasing noise amplitude.



$$x_{n+1} = f(x_n) + \xi_n$$



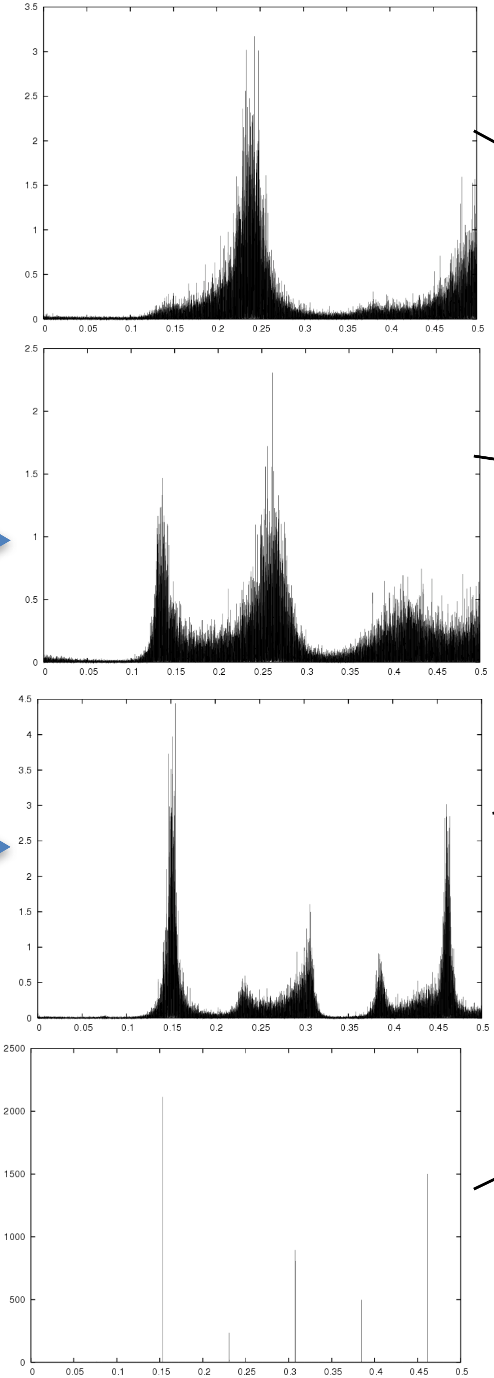
BZ map + uniform noise

[YS, 2009]

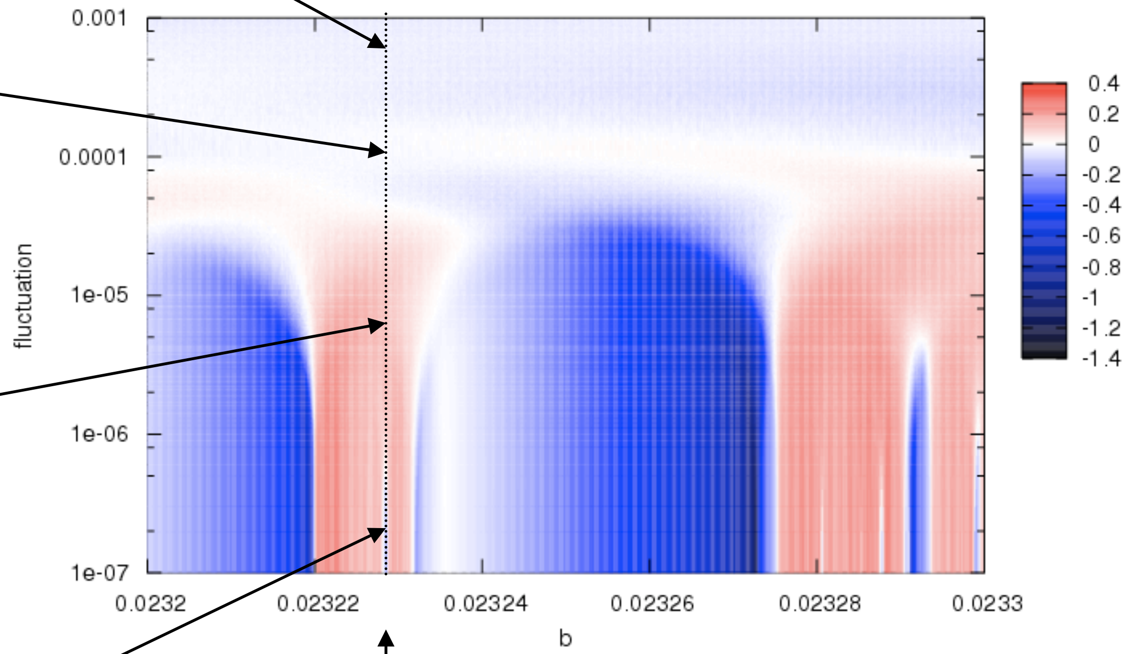
Power spectra

Chaotic

Chaotic



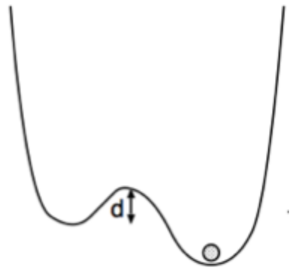
Lyapunov exponents



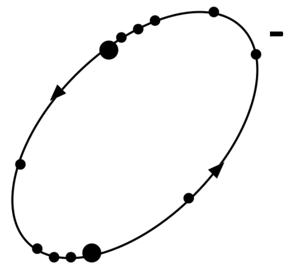
Both NIC and NIO are observed with different noise amplitude. [Sato, 2009]

Noise-induced phenomena

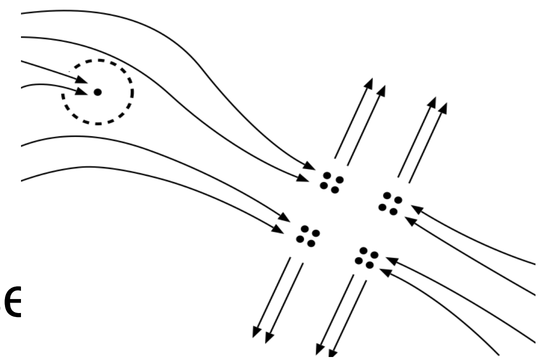
- Stochastic resonance
  [Benzi et. al., 1982]
  - Gradient dynamics
  - Potential barriers interact with noise



- Noise-induced synchronization
  [Teramae and Tanaka, 2004]
  - Oscillatory dynamics
  - Stagnation points in phase interact with noise



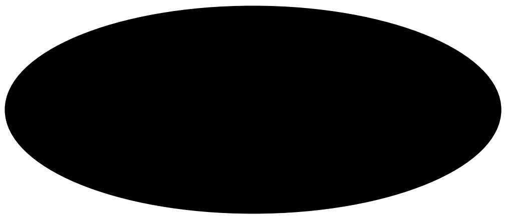
- Noise-induced chaos
  [G. Mayer-Kress and H. Haken, 1981]
  - Chaotic dynamics
  - Chaotic saddles, UPOs, … , interact with noise



Random dynamical systems

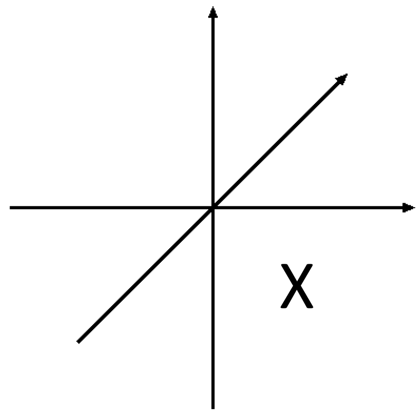
A random dynamical system is the combination of two systems (θ, ϕ).

Model of noise θ



Random influence $\longrightarrow$

Model of dynamics ϕ



Ω:{ω=(…,ω_0, ω_1, ω_2, …)}

$$\boxed{x_{n+1} = f(x_n) + \omega_n}$$

ω_n: noise

State space

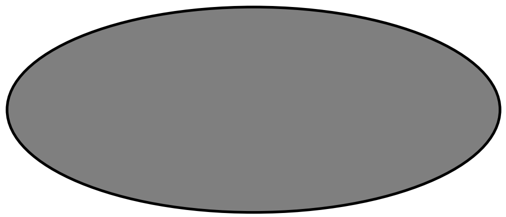
or $(\omega_n, x_n) = (\theta^n \omega_0,\ \phi(n, \theta^n \omega_0) x_0)$

Ω × X

Non-autonomous dynamical systems

A non-autonomous dynamical system is the combination of two systems (ψ, ϕ).

Model of environment ψ



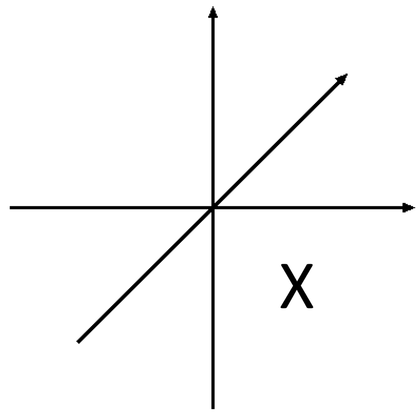
Influence

I: {i=(…,i0, i1, i2, …)}

$$\boxed{x_{n+1} = f(x_n) + I_n}$$

{I_n}: arbitrary input

Model of dynamics ϕ



State space

I × X

Random attractor and its stability

Random attractor: A(ω)

An invariant random set of $x_{n+1} = f(x_n) + \xi_n = \phi(n, \omega)x_0$

satisfies $\lim_{n\to\infty} d(\phi(n, \theta^n\omega)B, A(\omega)) = 0$

for a bounded set B ⊂ X.

Random Lyapunov exponent: λ(ω)

$$\lambda(\omega, x) = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial \phi(n, \omega)x}{\partial x} \right| \quad (x \in A(\omega))$$

We may use <λ> to measure average stability

Example: random point attractor

Langevin equation for Ornstein-Uhlenbeck process

$$dx = -\lambda x dt + \sigma dW_t \;\; (\lambda, \sigma > 0,\, W_t : Wiener\; process)$$

Random point attractor: $x(\omega)$

Invariant density: $\rho\big(x(\omega)\big) \sim \sqrt{\lambda/\pi\sigma^2}\exp\left(-\frac{\lambda x^2}{\sigma^2}\right)$

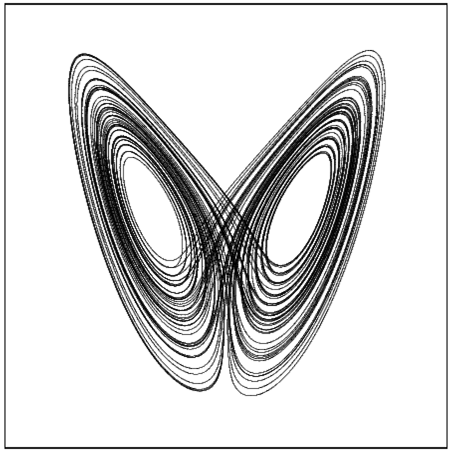
Lyapunov exponent: $-\lambda$

Example: random strange attractor

Lorenz system

$$dx/dt = s(y - x)$$
$$dy/dt = rx - y - xz$$
$$dz/dt = -bz + xy$$

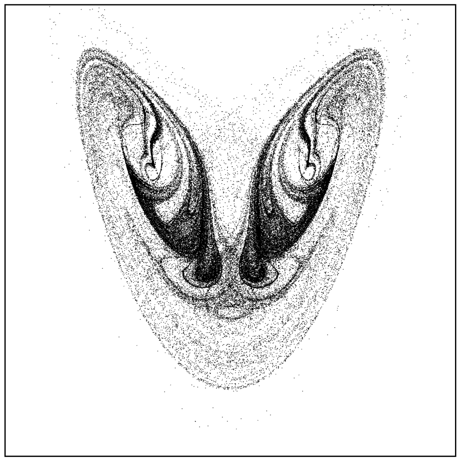
r = 28, s = 10, b = 8/3



Stochastic Lorenz system

$$dx = s(y - x)dt + \sigma x dW_t$$
$$dy = (rx - y - xz)dt + \sigma y dW_t$$
$$dz = (-bz + xy)dt + \sigma dW_t$$

r = 28, s = 10, b = 8/3 ,
σ = 0.3, Wt: Wiener process



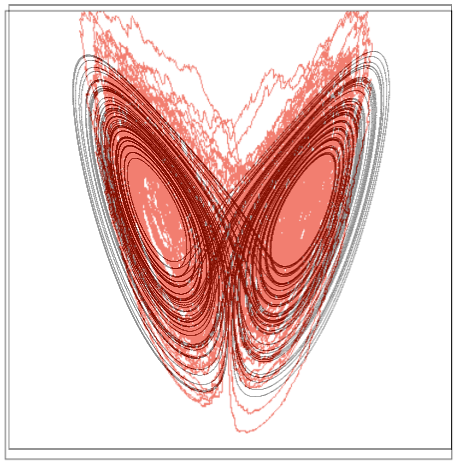
Example: Random strange attractor

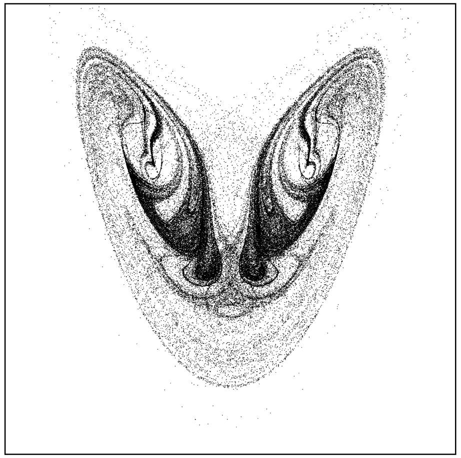
$$dx/dt = s(y - x)$$
$$dy/dt = rx - y - xz$$
$$dz/dt = -bz + xy$$

r = 28, s = 10, b = 8/3

$$dx = s(y - x)dt + \sigma x dW_t$$
$$dy = (rx - y - xz)dt + \sigma y dW_t$$
$$dz = (-bz + xy)dt + \sigma dW_t$$

r = 28, s = 10, b = 8/3 ,
σ = 0.3, Wt: Wiener process

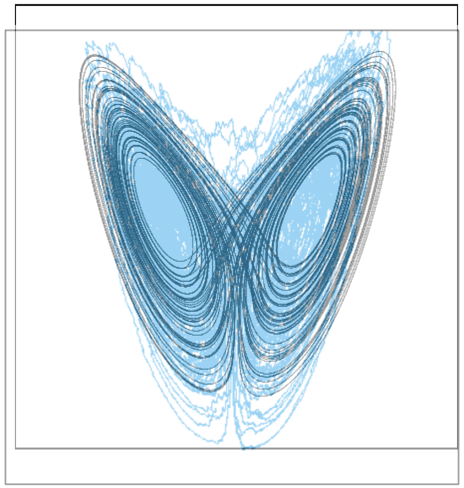




Example: Random strange attractor

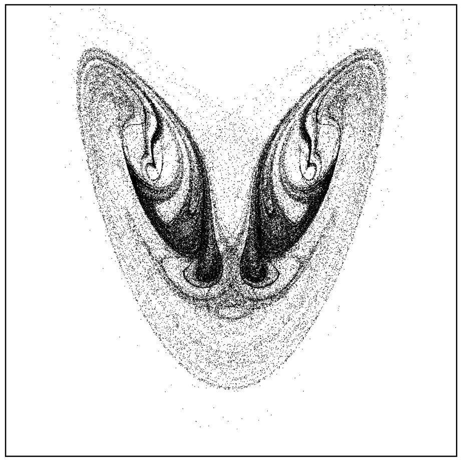
$$
\begin{aligned}
dx/dt &= s(y-x)\\
dy/dt &= rx - y - xz\\
dz/dt &= -bz + xy
\end{aligned}
$$

r = 28, s = 10, b = 8/3



$$
\begin{aligned}
dx &= s(y-x)dt + \sigma x dW_t\\
dy &= (rx - y - xz)dt + \sigma y dW_t\\
dz &= (-bz + xy)dt + \sigma dW_t
\end{aligned}
$$

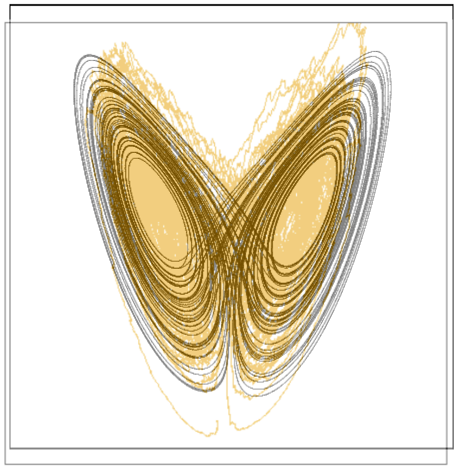
r = 28, s = 10, b = 8/3 ,
σ = 0.3, Wt: Wiener process



Example: Random strange attractor

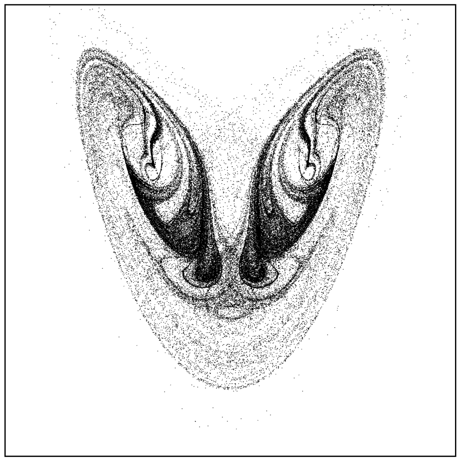
$$dx/dt = s(y - x)$$
$$dy/dt = rx - y - xz$$
$$dz/dt = -bz + xy$$

r = 28, s = 10, b = 8/3



$$dx = s(y - x)dt + \sigma x dW_t$$
$$dy = (rx - y - xz)dt + \sigma y dW_t$$
$$dz = (-bz + xy)dt + \sigma dW_t$$

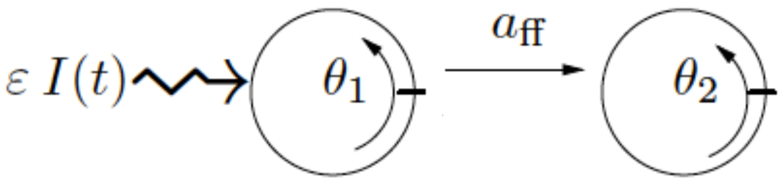
r = 28, s = 10, b = 8/3 ,
σ = 0.3, Wt: Wiener process



Noise-induced phenomena in random dynamical systems

- Noise-induced phenomena in orbits
  - Noise-induced chaos, noise-induced order, noise-induced synchronization, …
- Noise-induced phenomena in densities
  - Stochastic resonance, stochastic stability, statistical periodicity, …
- Noise-induced phenomena in basins
  - Noise-induced riddling, noise-induced reproducibility,…

Stochastic coupled oscillators



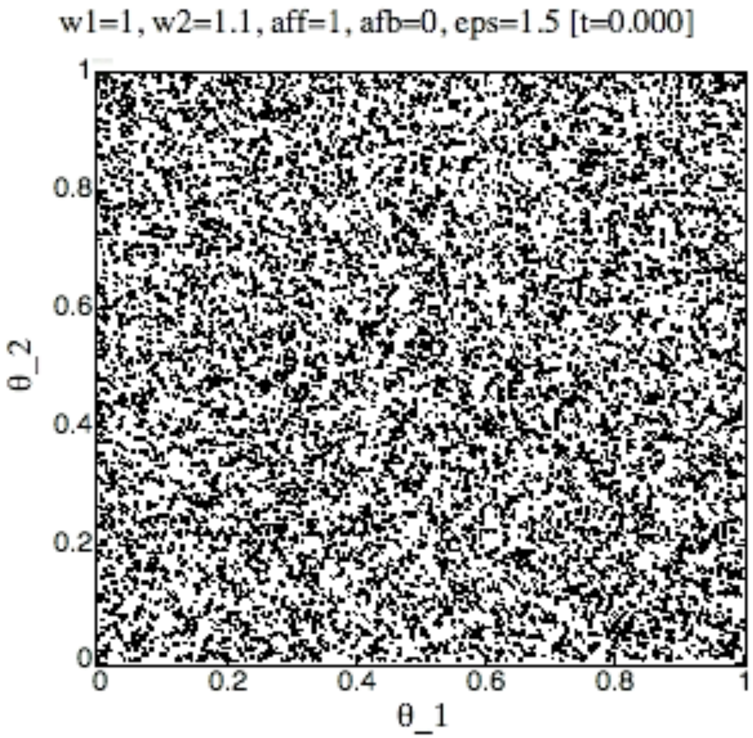
$$\dot{\theta}_1 = \omega_1 \qquad\qquad\qquad\qquad + \varepsilon z(\theta_1)\, I(t)\,,$$
$$\dot{\theta}_2 = \omega_2 + a_{\mathrm{ff}}\, z(\theta_2)\, g(\theta_1)\,.$$

$$z(\theta) = \tfrac{1}{2\pi}(1 - \cos(2\pi\theta))$$

$$\int_{-b}^{b} g(\theta)\, d\theta = 1$$

I(t): White Gaussian

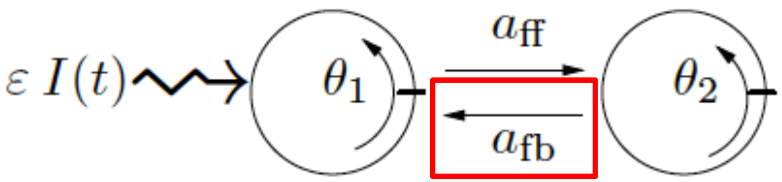
Video by K. Lin



Random point attractor and noise-induced synchronization

[K. Lin, L-S. Young, 2008]

Stochastic coupled oscillators



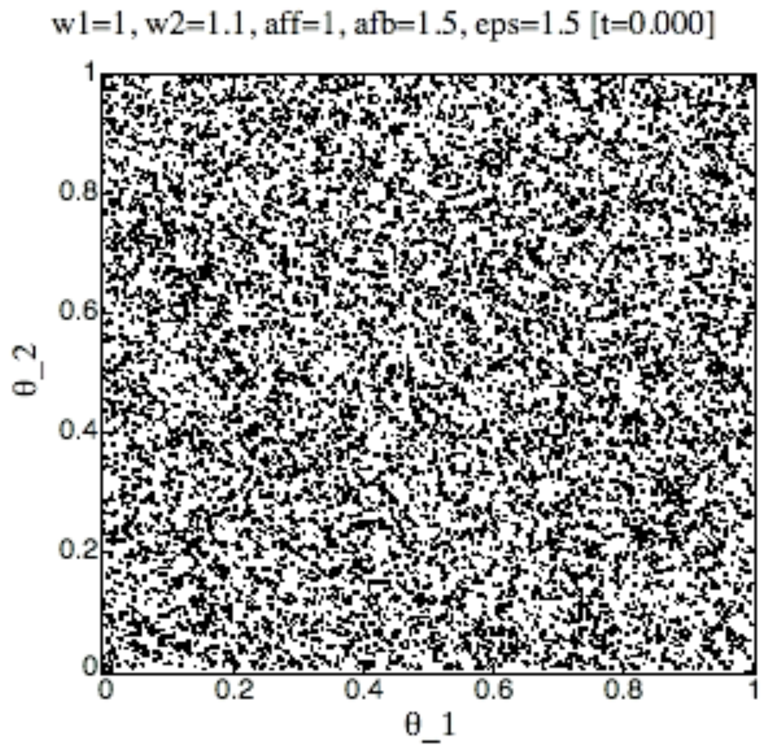
$$\dot{\theta}_1 = \omega_1 + a_{\text{fb}}\, z(\theta_1)\, g(\theta_2) + \varepsilon z(\theta_1)\, I(t)\,,$$
$$\dot{\theta}_2 = \omega_2 + a_{\text{ff}}\, z(\theta_2)\, g(\theta_1)\,.$$

$$z(\theta) = \frac{1}{2\pi}(1 - \cos(2\pi\theta))$$

$$\int_{-b}^{b} g(\theta)\, d\theta = 1$$

I(t): White Gaussian

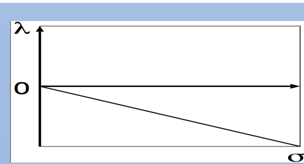
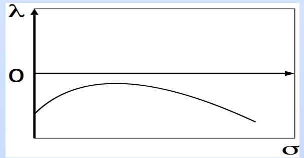
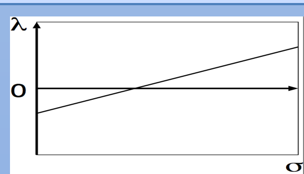
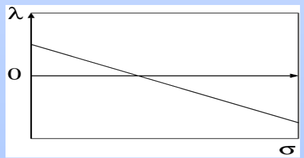
Video by K. Lin



Random strange attractor and stochastic chaos (noise-induced filamentation).

[K. Lin, L-S. Young, 2008]

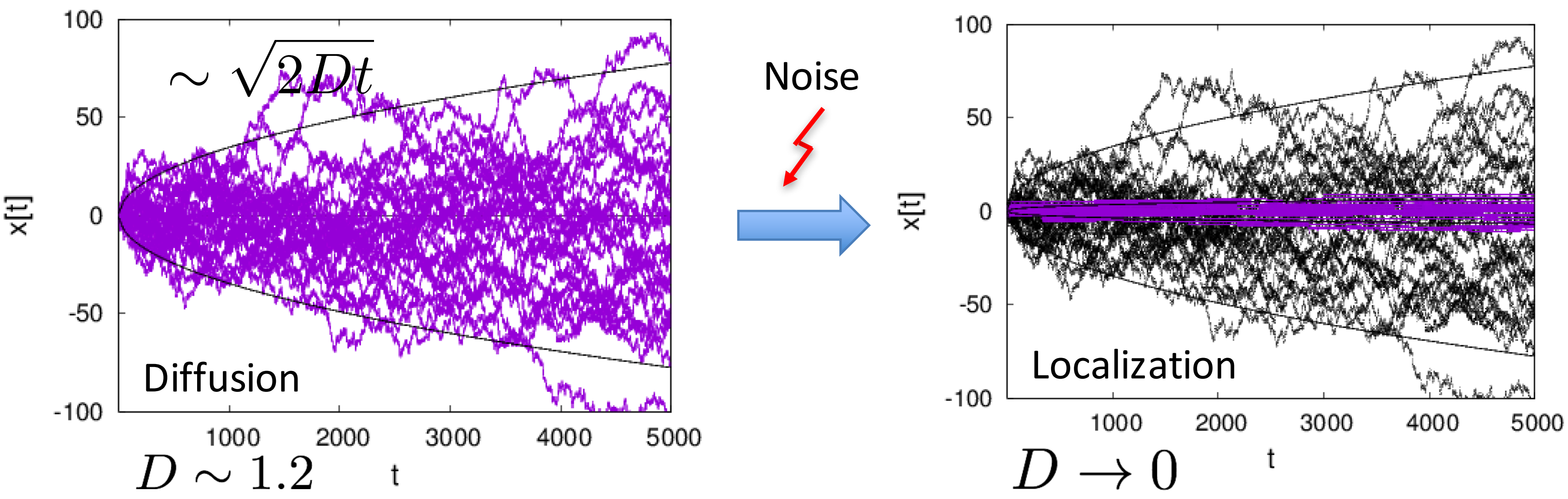
Random dynamical systems analysis for noise-induced phenomena

Noise-induced phenomena	Stationary state	Topological bifurcation	Top Lyapunov exponent λ vs noise amplitude σ
Noise-induced synchronization	random point attractor	Yes	
Stochastic resonance	random periodic attractor	No	
Noise-induced chaos	random strange attractor	Yes	
Noise-induced order	“window phenomena”	No	
Noise-induced intermittency	non-stationary (infinite ergodic)	Not at onset of topological bifurcation	$\sigma=\sigma^*$, $\lambda=0$

[A. Cherubini, YS, M. Rasmussen, J. Lamb, 2017]
[YS, T-S Doan, M, Rasmussen, J. Lamb, to be submitted] [YS, R. Klages, to be submitted]

Noise-induced transition in open dynamics

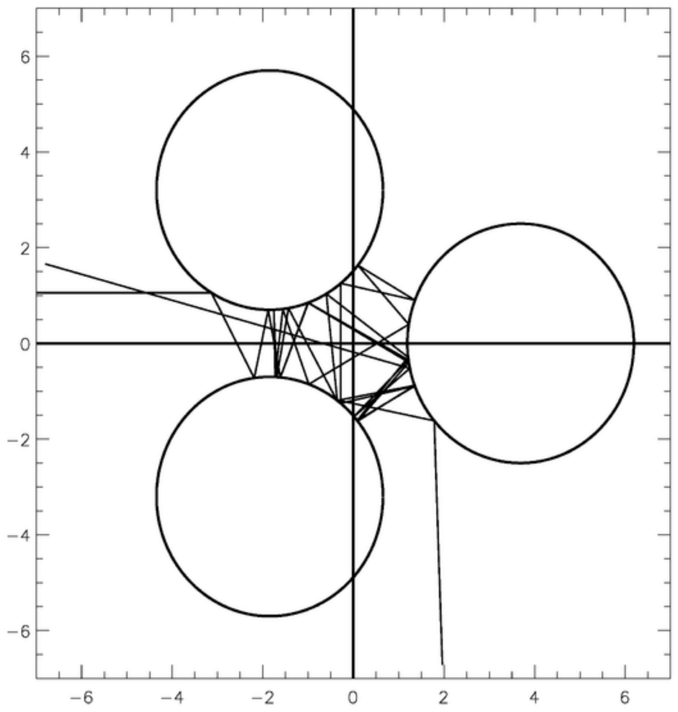
Anomalous diffusion in random dynamical systems
[Collaboration with Rainer Klages
at Queen Mary University of London, UK]



2. Deterministic diffusion

Deterministic diffusion

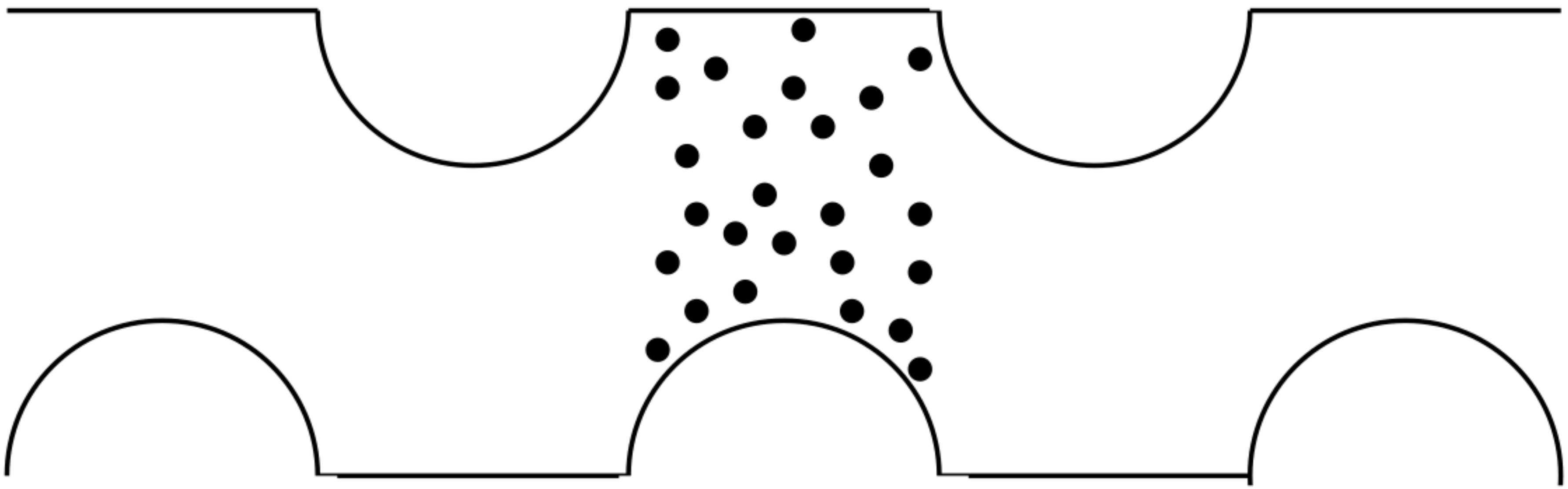
Chaotic scattering



Gaspard–Rice scattering

Deterministic diffusion

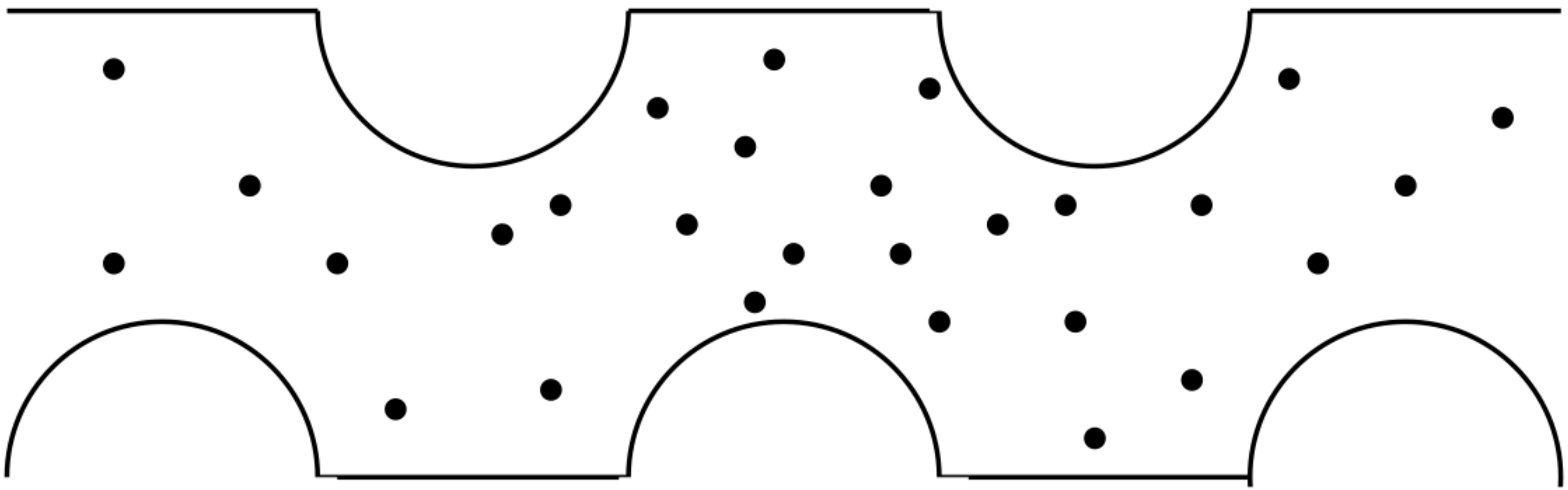
Open billiard



Periodic Lorenz gas

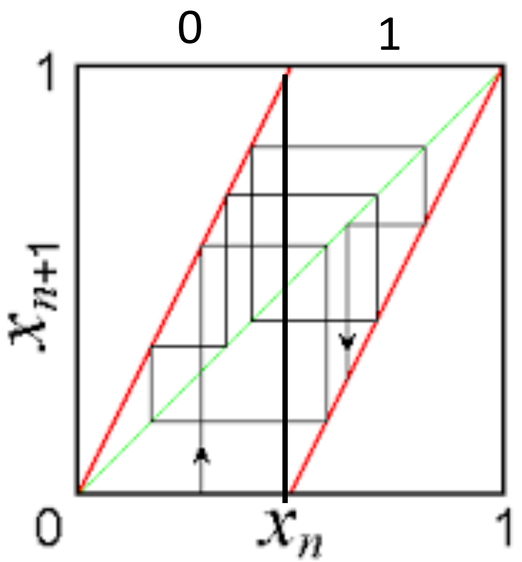
Deterministic diffusion

Open billiard



Lorenz gas

Bernoulli map and coin tossing



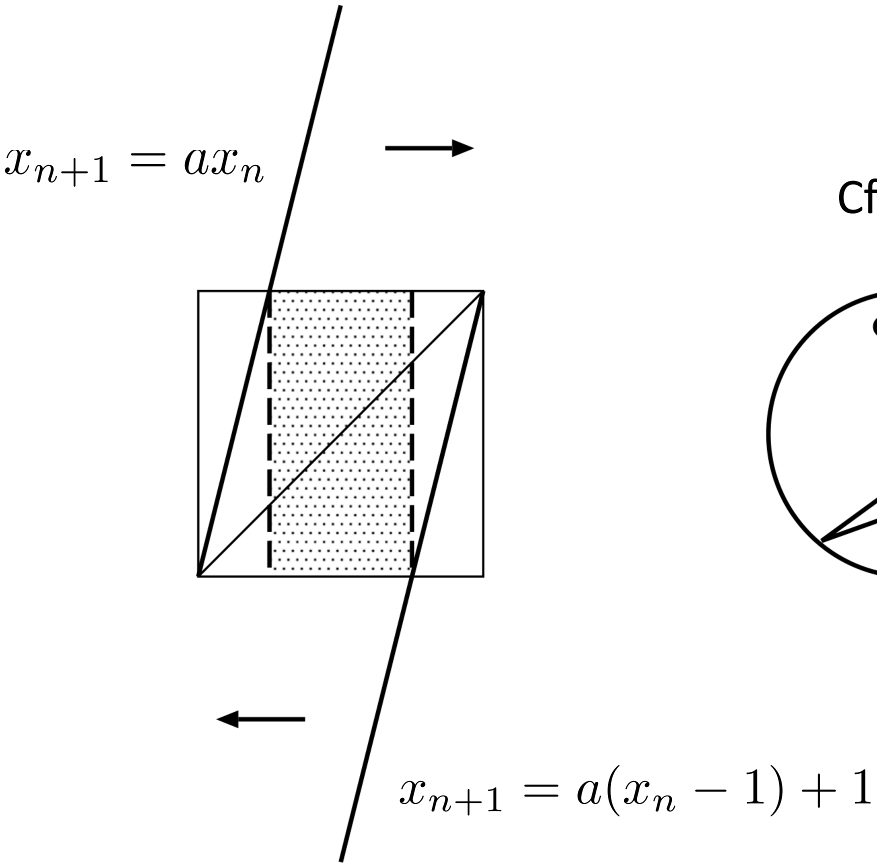
$$B_a(x) = ax \pmod{1}$$

a=2

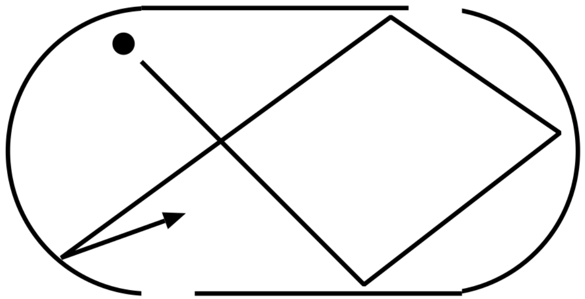
“Coarse-grained” chaotic dynamics

Coin tossing

Open Bernoulli map

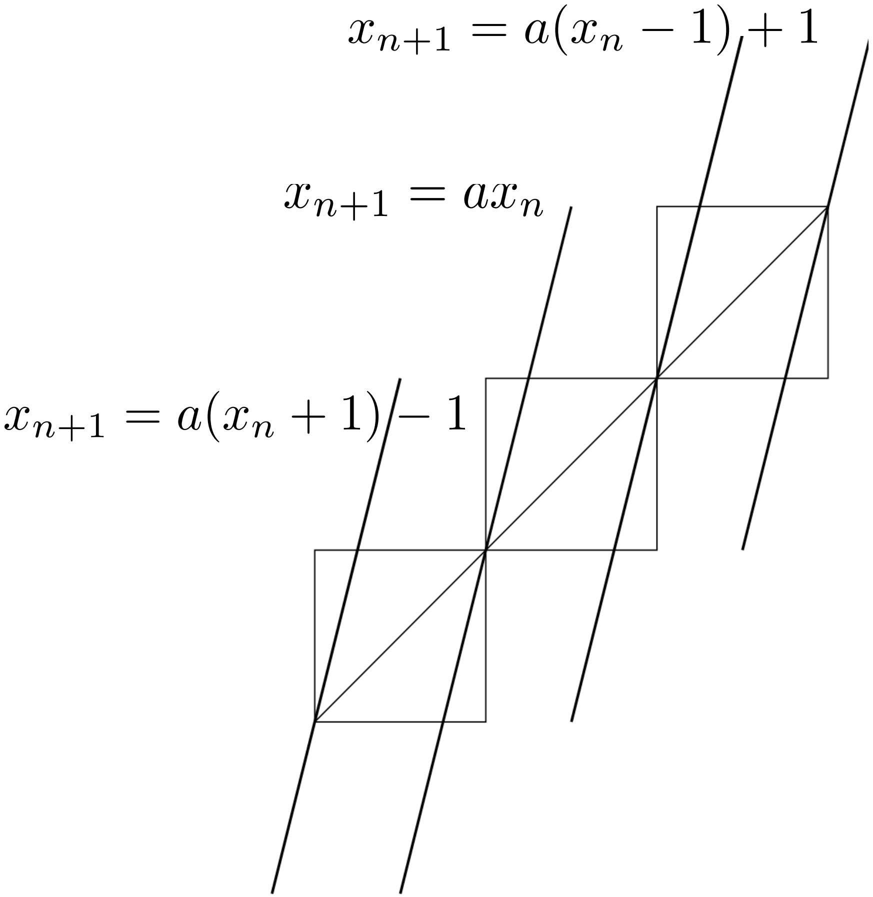


Cf. Open billiard (schematic view)



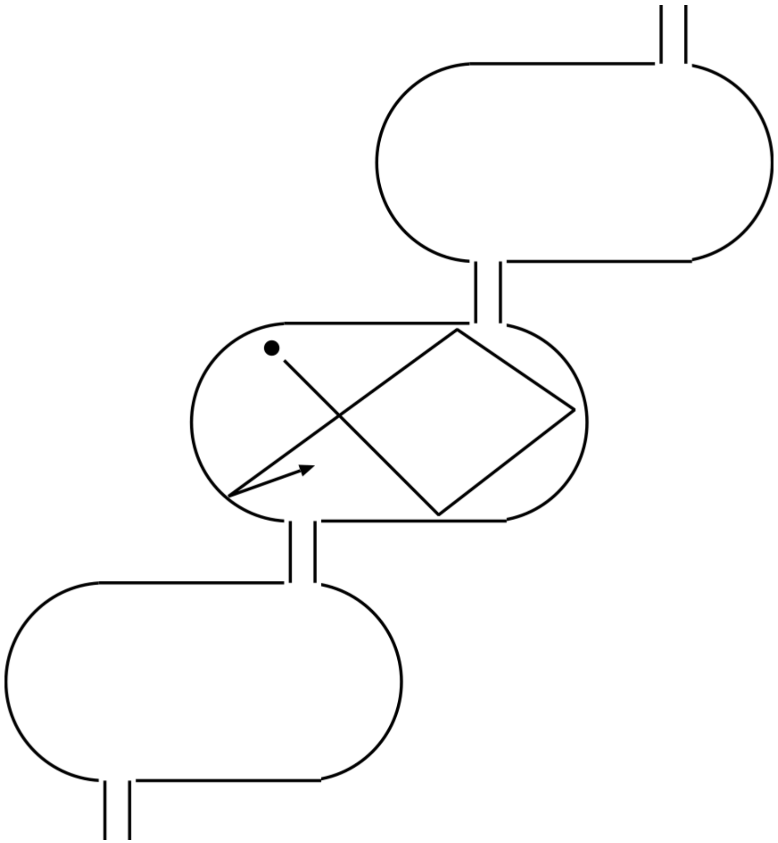
$$B_a(x) = ax \ (\mathrm{mod} 1) \quad a>2$$

Open Bernoulli map



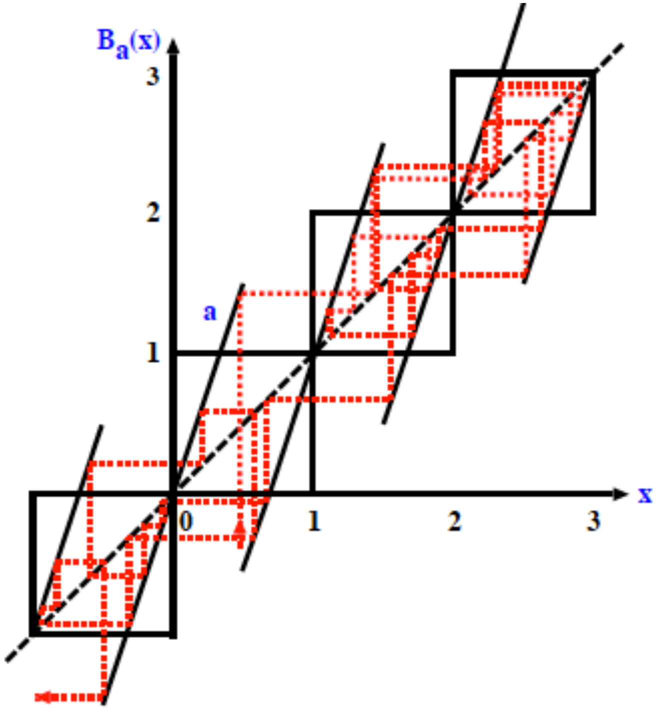
$$B_a(x+1) = B_a(x) + 1$$

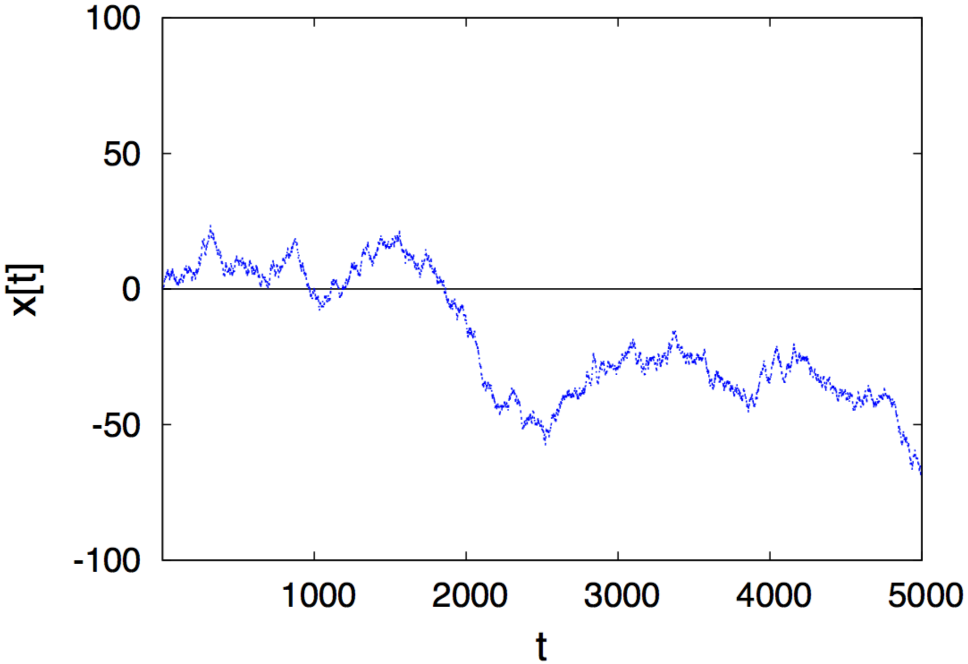
$$B_a(x) = ax \ (\mathrm{mod} 1)$$ a>2



Cf. Chain of open billiard

Open Bernoulli map and random walk

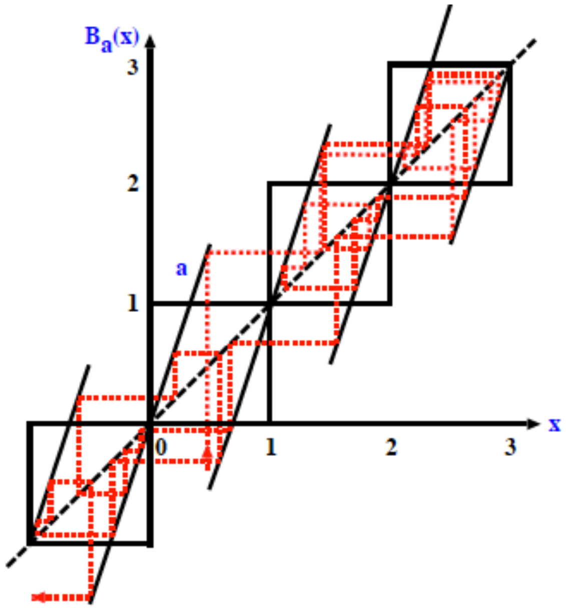


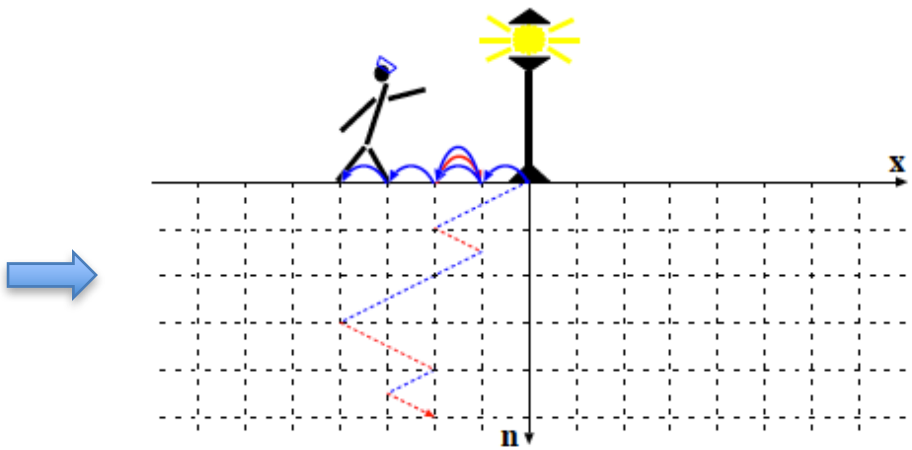


$$B_a(x+1) = B_a(x) + 1$$

$$B_a(x) = ax \ (\mathrm{mod} 1)$$ a>2

Open Bernoulli map and random walk



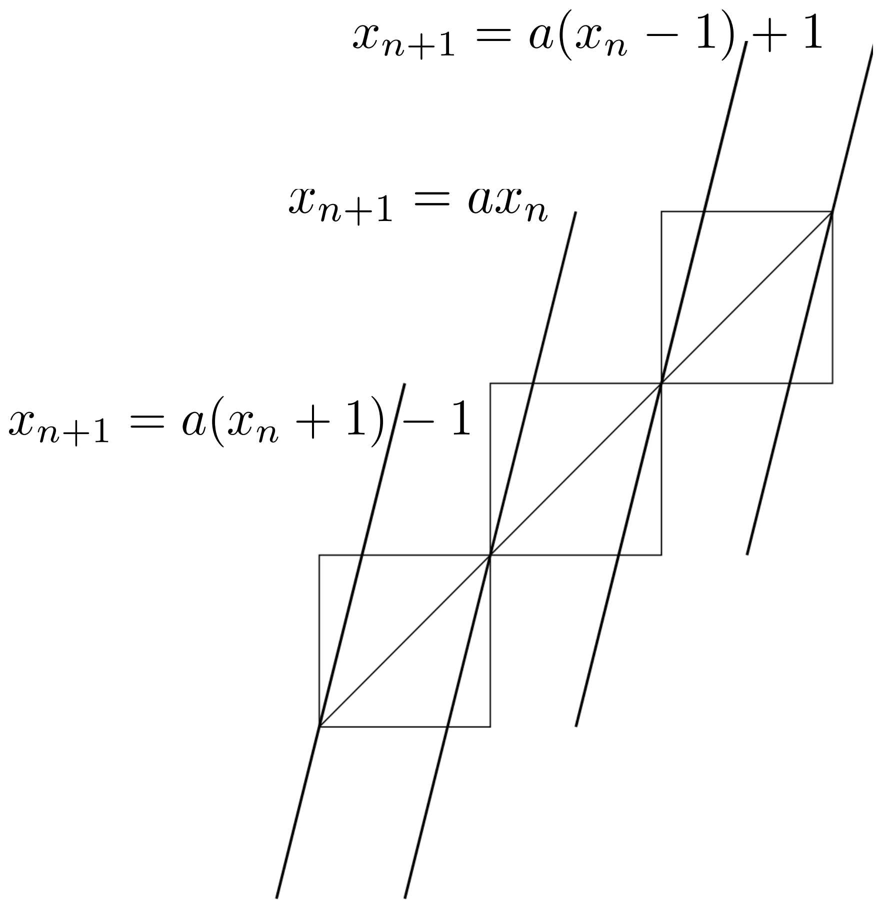


Dynamics of open Bernoulli map

Random walk

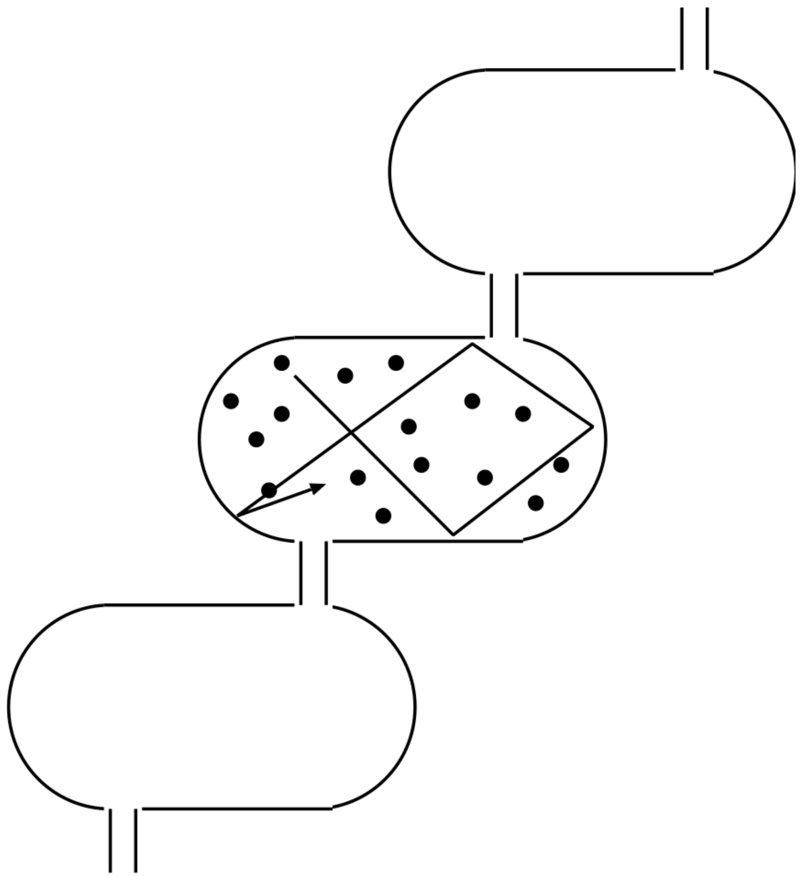
[Figures from Klages 95]

Open Bernoulli map and diffusion



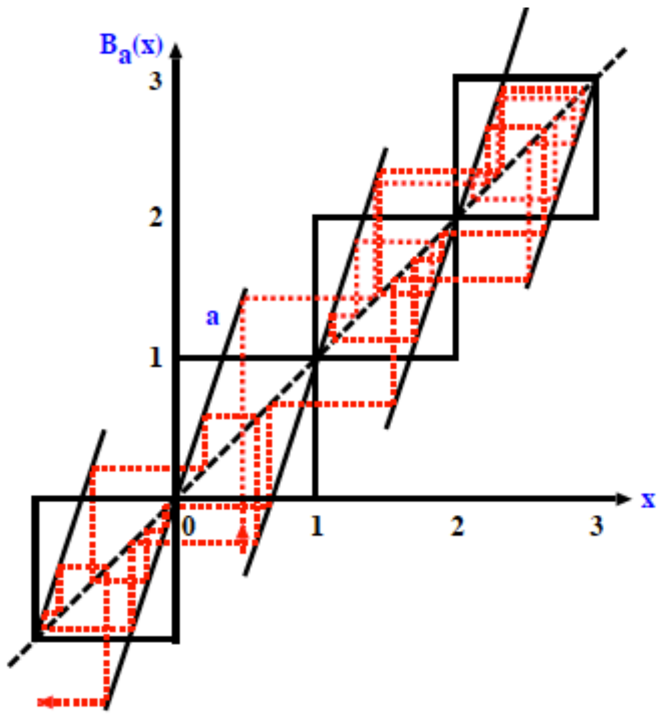
$$B_a(x+1) = B_a(x) + 1$$

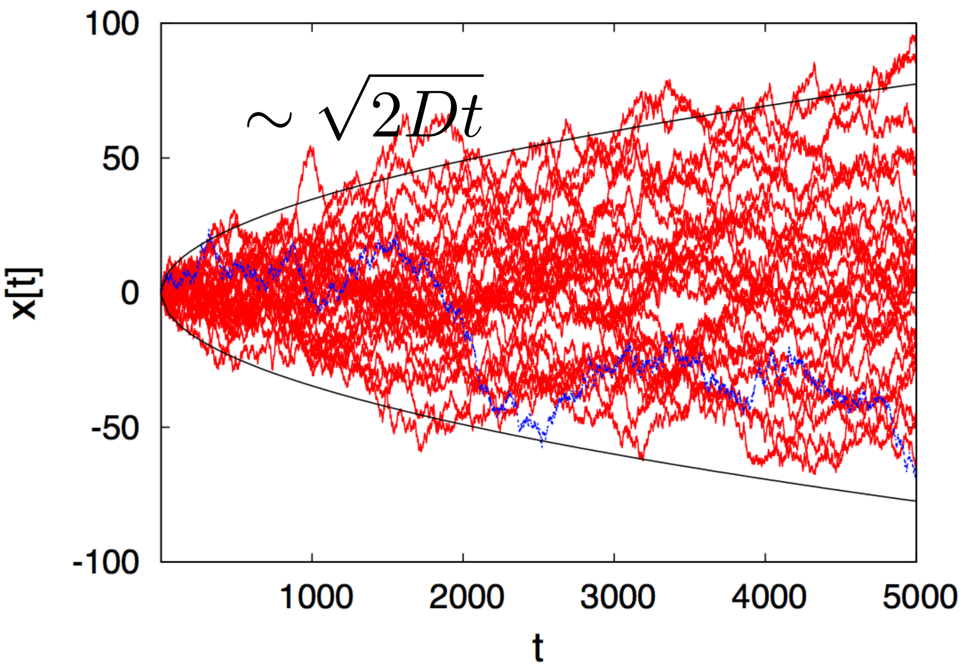
$$B_a(x) = ax \pmod 1$$ a>2



Cf. Chain of open stadium billiard with multiple particles

Open Bernoulli map and diffusion





$$B_a(x+1) = B_a(x) + 1$$

$$B_a(x) = ax \pmod{1}$$ a>2

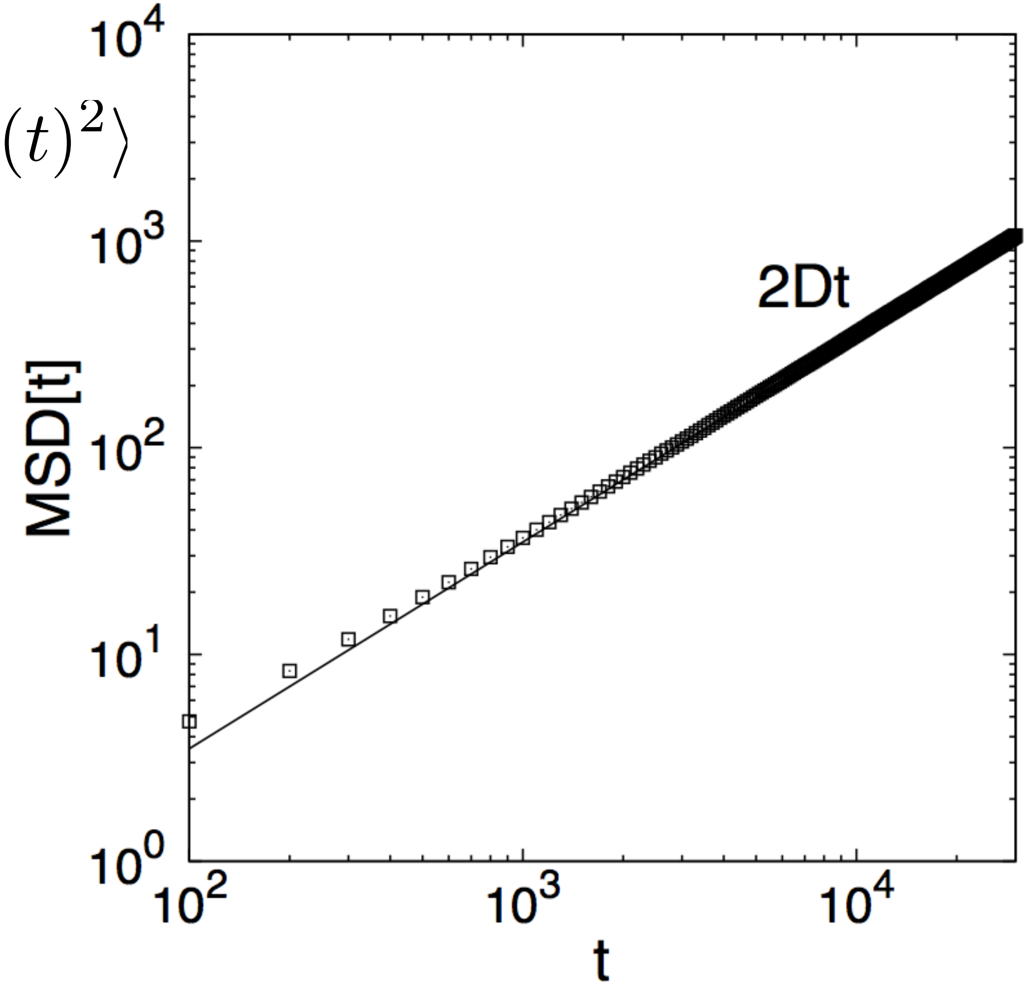
$$\frac{\partial P}{\partial t} = D\frac{\partial^2 P}{\partial x^2}$$

Equation of motion of sample measure

～ Diffusion equation

Mean square displacement

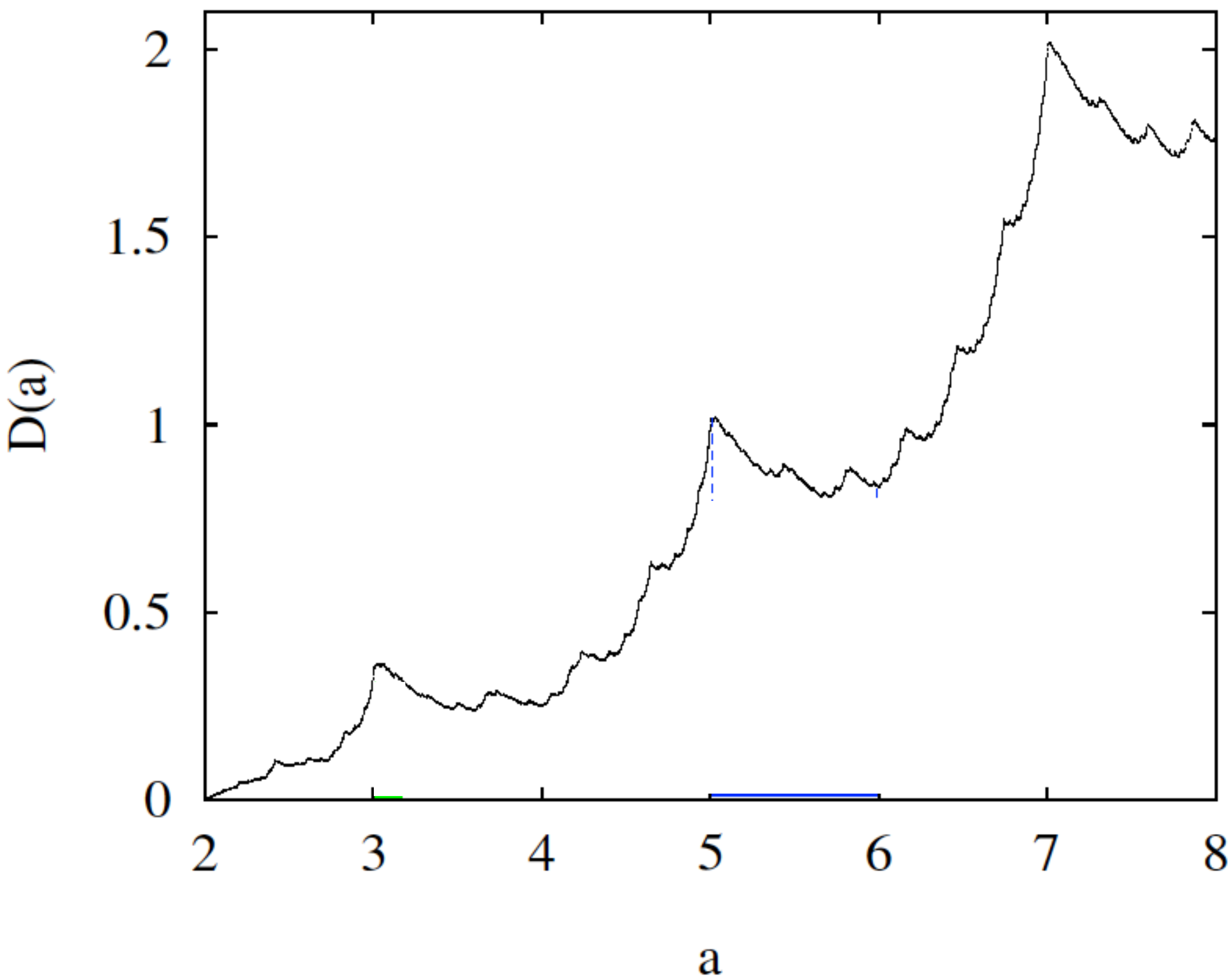
$\langle x(t)^2 \rangle$



$$D = \lim_{t \to \infty} \frac{\langle x(t)^2 \rangle}{2t}$$

$$\langle x(t)^2 \rangle \sim t$$

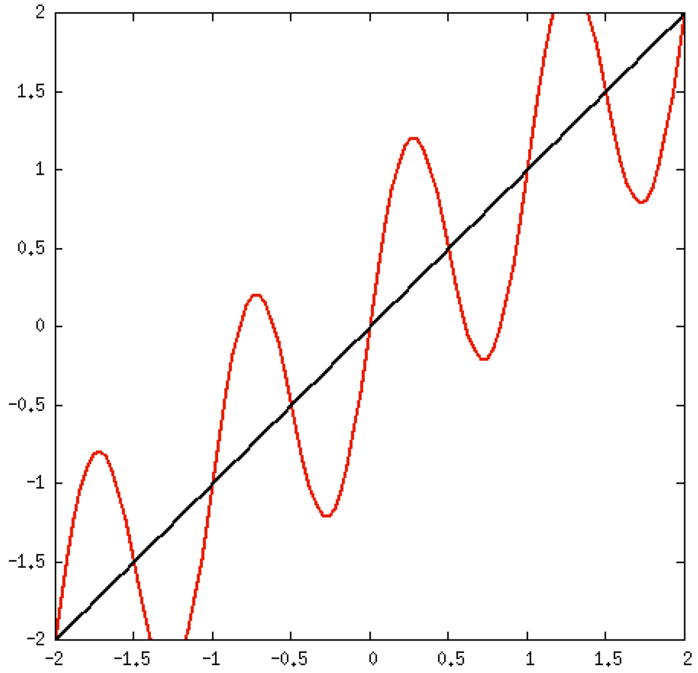
Diffusion coefficient and expansion rate



$$D = \lim_{t \to \infty} \frac{\langle x(t)^2 \rangle}{2t}$$

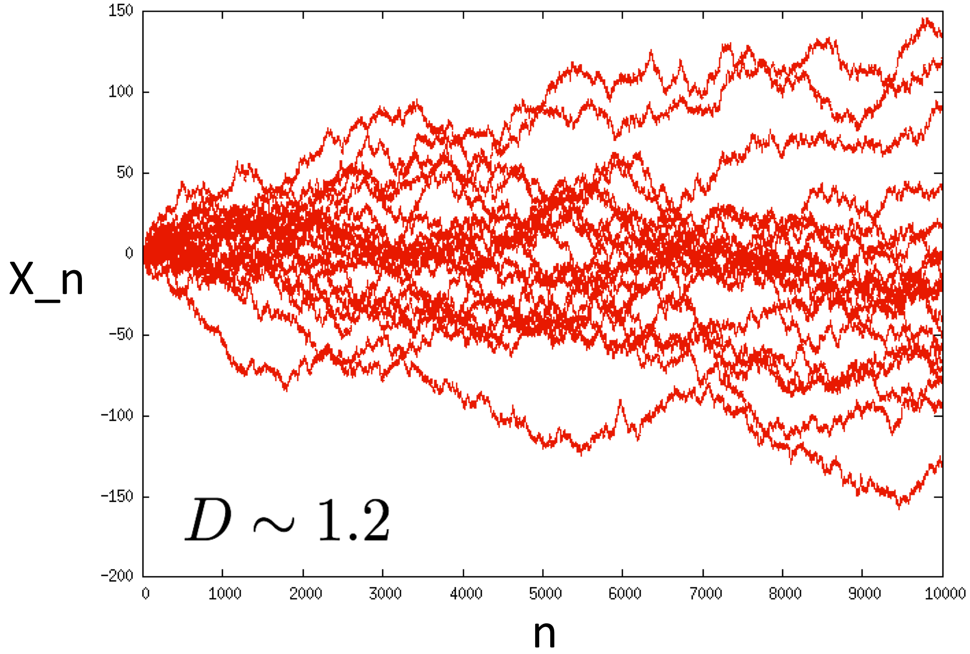
[Klages 95]

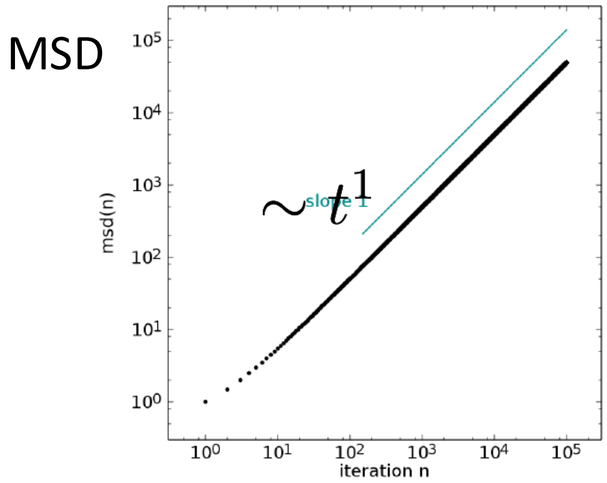
Climbing sine map and diffusion



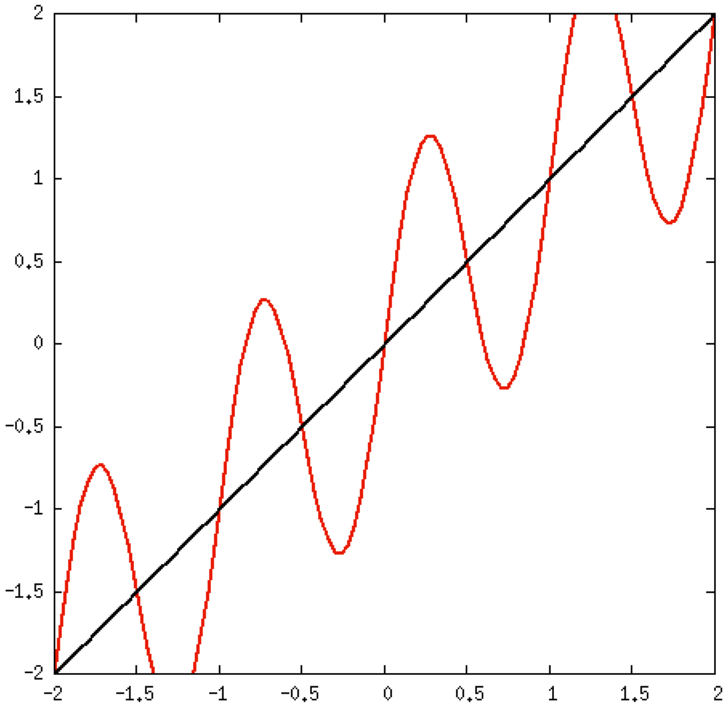
$$x_{n+1} = x_n + a\sin(2\pi x_n)$$

a=0.95



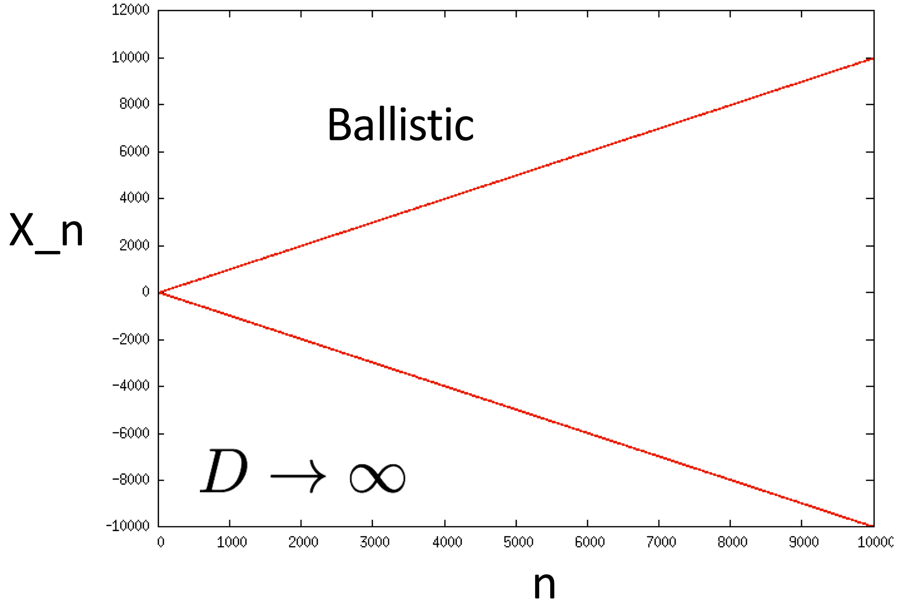


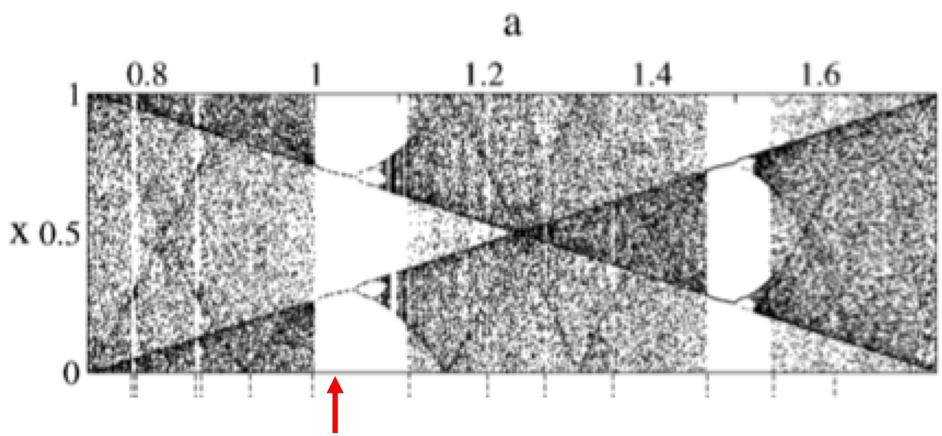
Climbing sine map and diffusion



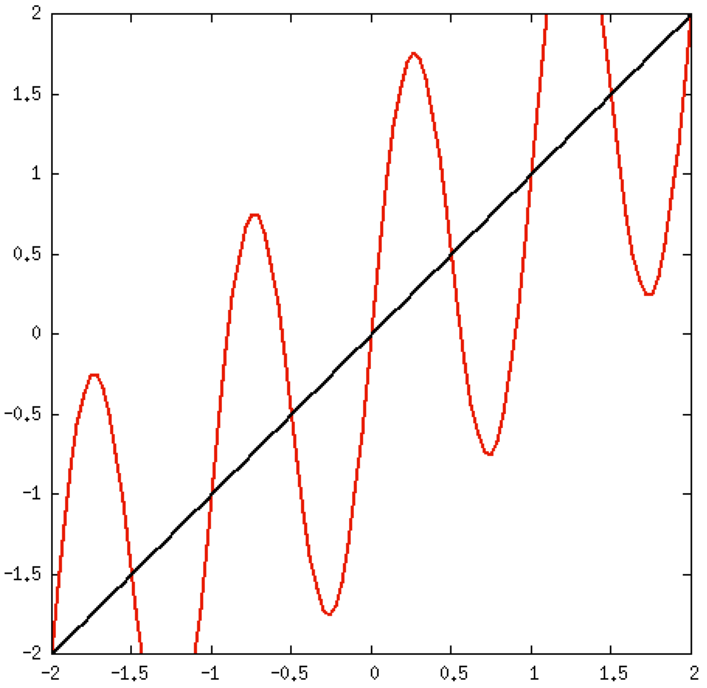
$$x_{n+1} = x_n + a\sin(2\pi x_n)$$

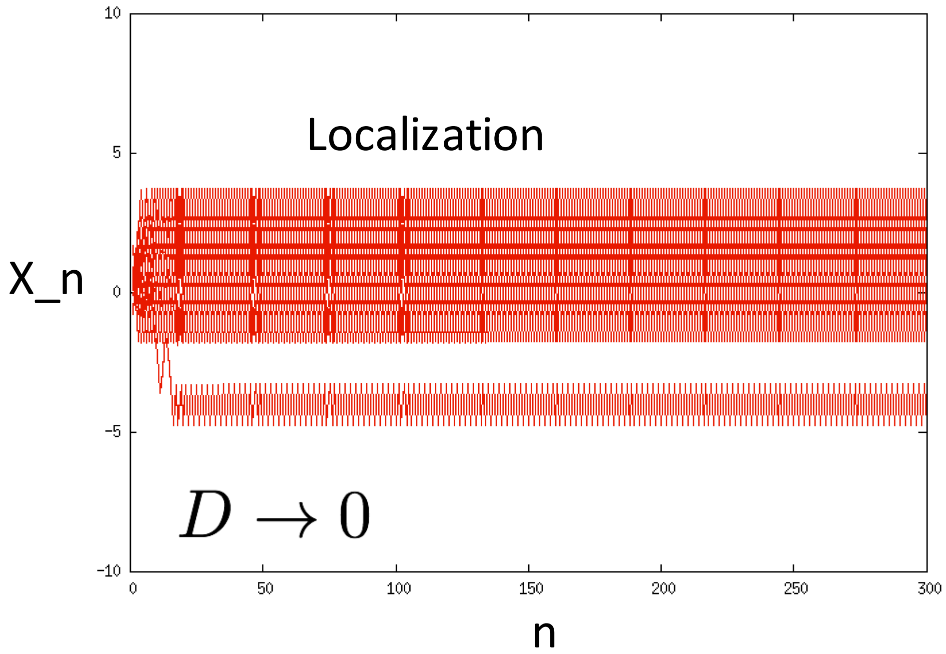
a=1.01





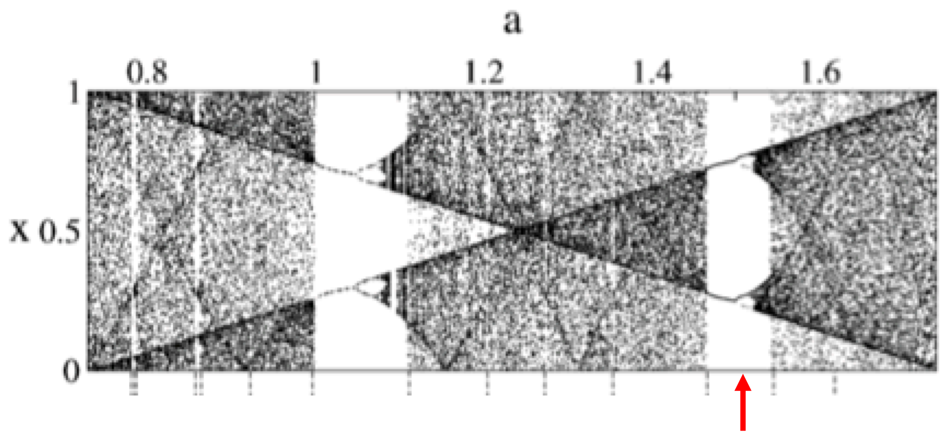
Climbing sine map and diffusion



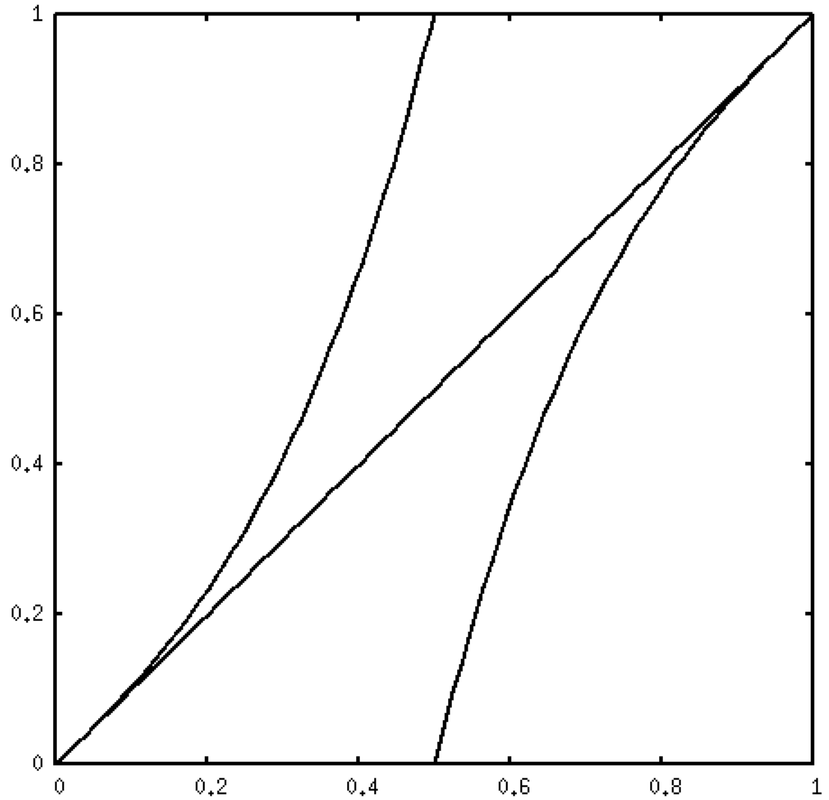


$$x_{n+1} = x_n + a\sin(2\pi x_n)$$

a=1.5

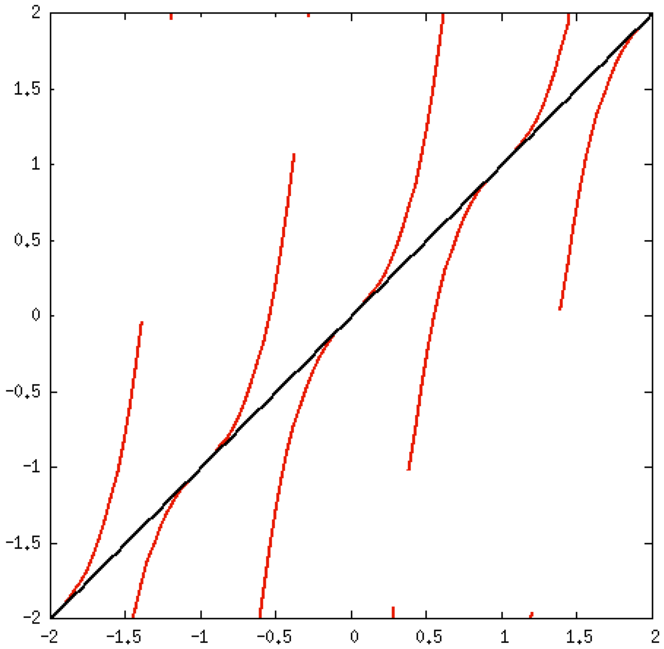


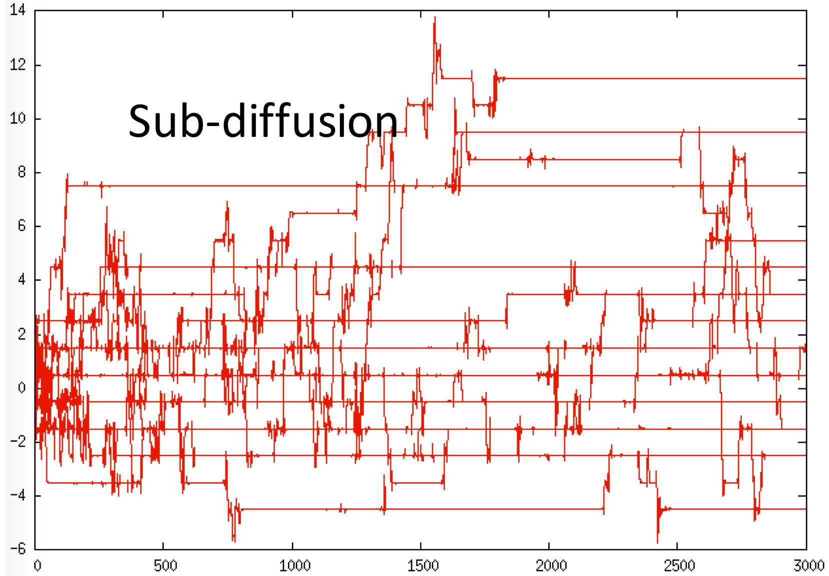
Pomeau-Manneville map

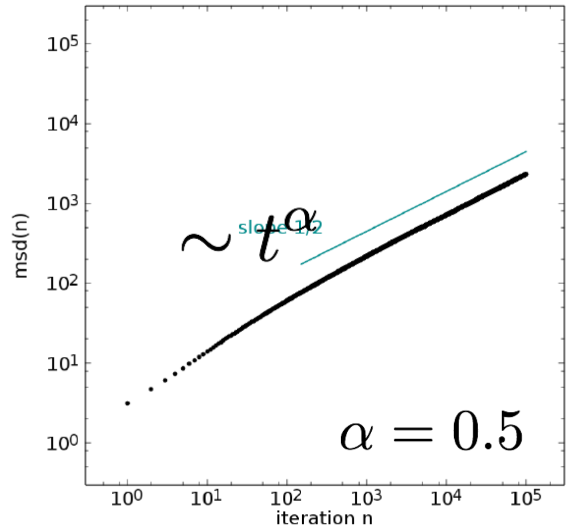


$$x_{n+1} = \begin{cases} x_n + (\frac{1}{2})^z x_n^z & x \in [0, 1/2) \\ x_n - (\frac{1}{2})^z (1 - x_n)^z & x \in [1/2, 1] \end{cases}$$

Open Pomeau-Manneville map and sub-diffusion







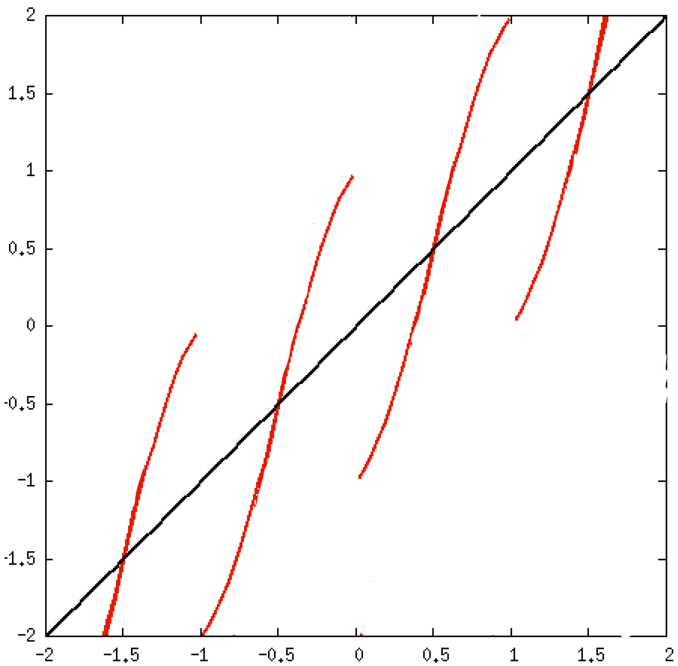
$$x_{n+1} = \begin{cases} x_n + a x_n^z & x \in [0, 1/2) \\ x_n - a(1 - x_n)^z & x \in [1/2, 1] \end{cases}$$

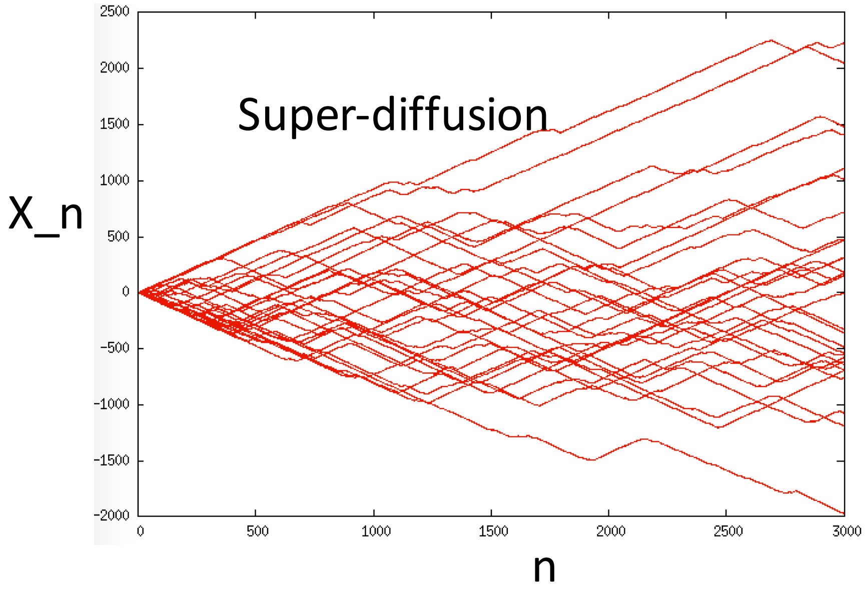
$$F(x+1) = F(x) + 1$$

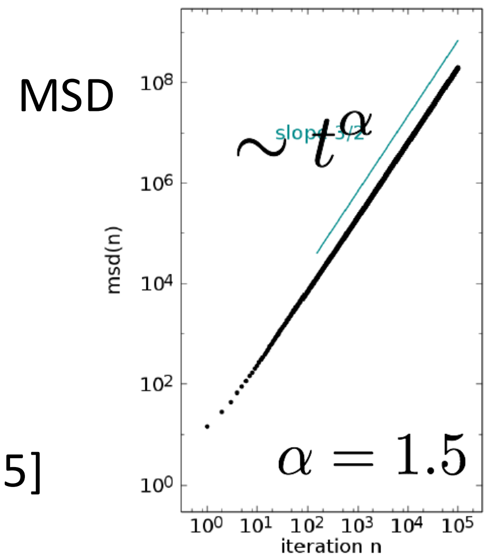
a=6, z=3

[Geisel et.al., 1984]

Open Pomeau-Manneville map and super-diffusion







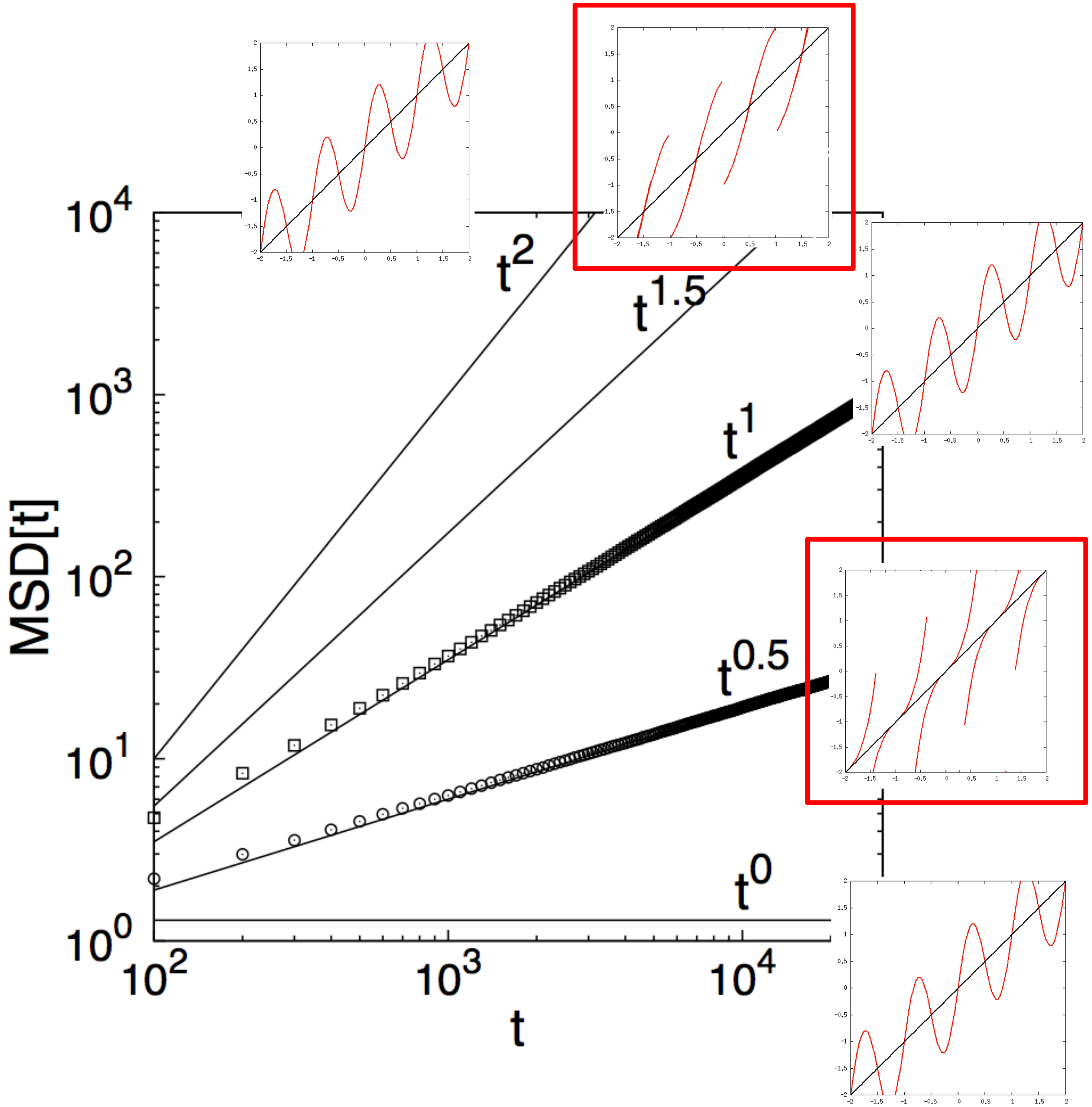
$$x_{n+1} = \begin{cases} x_n + a x_n^z - 1 & x_n \in [0, 1/2) \\ x_n - a(1 - x_n)^z + 1 & x_n \in [1/2, 1) \end{cases}$$

$$F(x+1) = F(x) + 1$$

a=6, z=5/3

[Geisel et.al., 1985]

Deterministic anomalous diffusion



$$D = \lim_{t\to\infty} \frac{\langle x(t)^2 \rangle}{2t}$$

$$\langle x(t)^2 \rangle \sim t^{\alpha}$$

α=2 : Ballistic

1<α<2 : Super-diffusion

α=1 : Normal diffusion

0<α<1 : Sub-duffusion

α=0 : Localization

3. Anomalous diffusion in random dynamical systems

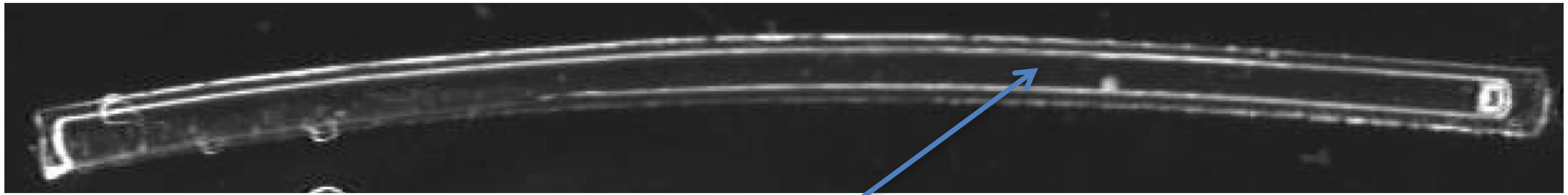
Summary

1. Climbing sine map may show noise-induced anomalous sub-diffusion.

2. Universality of intermittency in 1D random dynamical systems is different from those in deterministic 1D dynamical systems.]

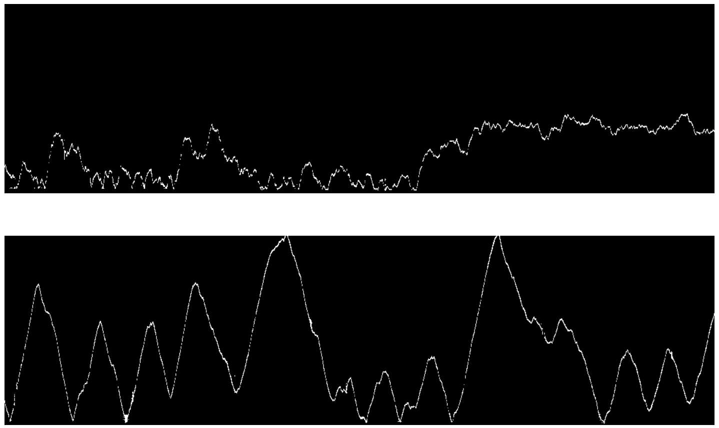
3. Weak ergodicity breaking caused by noise-induced synchronization.

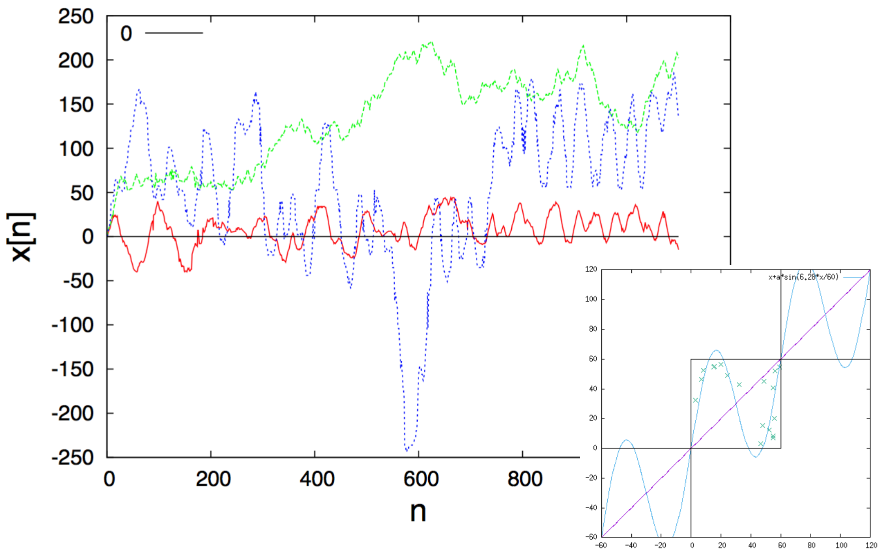
Application: Spatially extended RDS model for locomotion of microorganisms



Tetrahymena

Environment: 1d tube.

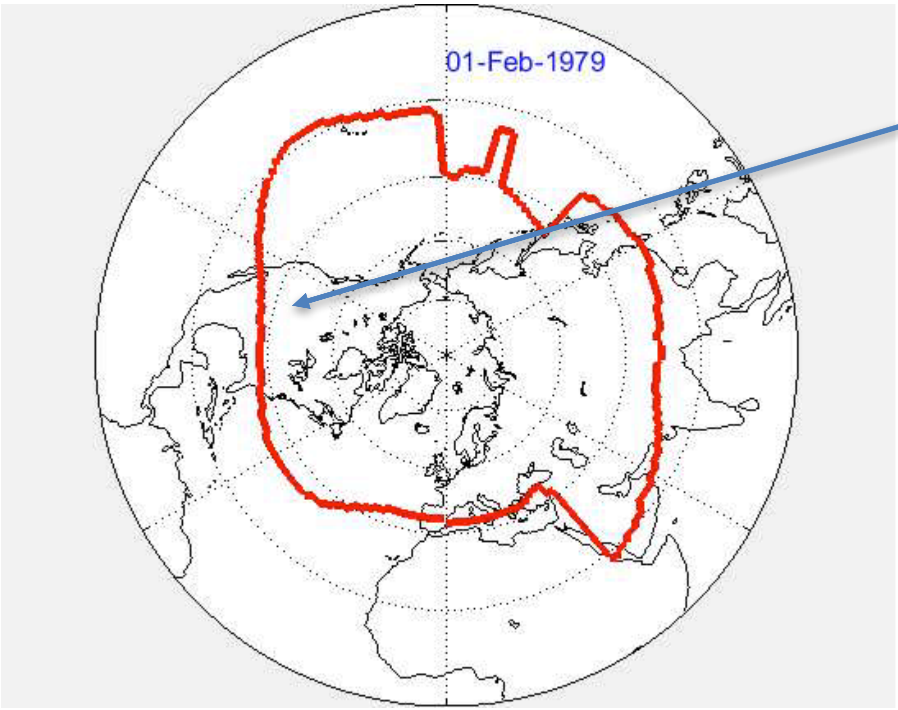


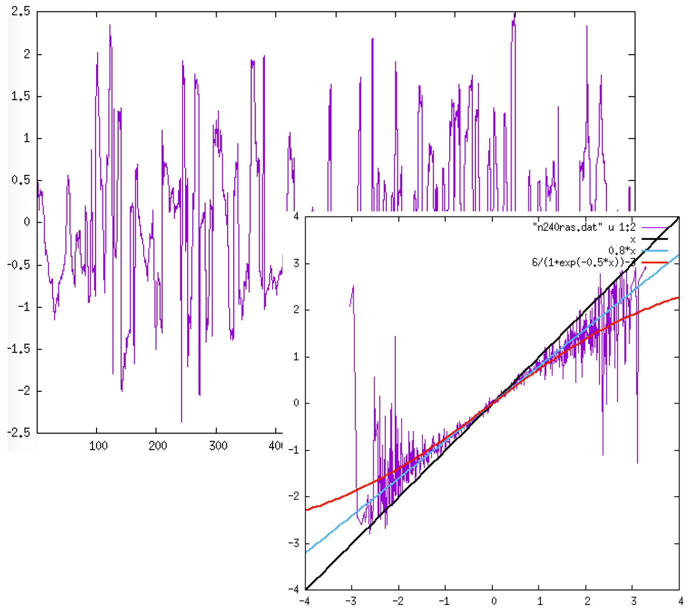


[YS, T. Nakagaki, in preparation]

Application: Coupled RDS model for atmospheric jet stream

Creation and annihilation of blocking phenomena





Jet latitude dynamics
at rocky mountains

[Our on-going research project at LSCE!]

Thank you