

# The form of steady states in transmission grids

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Réunion GT Climat - Méca Stat

# Transmission grid structure

AC grid =



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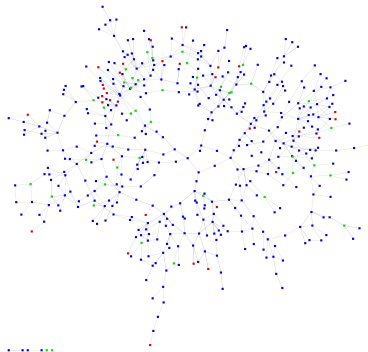
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essential for grid stability: global frequency synchronization

stabilize grid frequency:  $\langle \dot{\theta}_i \rangle = 50\text{Hz}$

minimize phase differences:  $|\theta_i - \theta_j| \ll 1$



# Kuramoto models

grid as generalized Kuramoto model (Filatrella et al 2008):

$$\alpha \ddot{\theta}_i + \gamma \dot{\theta}_i = \hat{\Omega}_i + \hat{P}_{max} \sum_{j=1}^N A_{ij} \sin(\theta_j - \theta_i)$$

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**In co-rotating frame:**

frequency synchronization  $\Rightarrow$  steady state

steady states, linear stability as in classic model (Doerfler et al 2010):

$$\dot{\theta}_i = \omega_i + \lambda \sum_{j=1}^N A_{ij} \sin(\theta_j - \theta_i)$$

# Research context

## *Physics literature:*

Given the **infinitely** large networks, **symmetric** frequency distributions and varying topologies, at which  $\lambda_C$  **onset** of synchronization?

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## *Missing:*

- analytic description of robustness against input fluctuations
- analytic description of frequency correlations
- modelling of full synchronization on non-trivial topologies

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## basic topologies:



all-to-all,  $k_i = 5$



random regular,  $k_i = 3$



random,  $\langle k \rangle = \frac{10}{3}$

## Naive MF model

**equivalence of all  $G$  generators ( $C$  consumers):**  $\theta_{G,C} = \langle \theta \rangle_{G,C}$

**single equation for  $\Delta\theta \equiv \langle \theta \rangle_G - \langle \theta \rangle_C$ :**  $\Delta\dot{\theta} = \frac{N}{C} - \lambda \langle k \rangle \sin [\Delta\theta]$

**fixed point:**  $\Delta\theta^* = \arcsin [\omega_G N / (\lambda \langle k \rangle C)]$

**stable for:**  $\lambda \geq \lambda_C = N / \langle k \rangle / C$

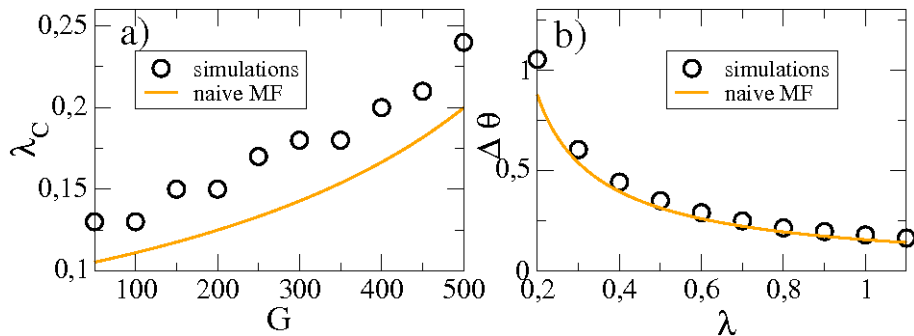
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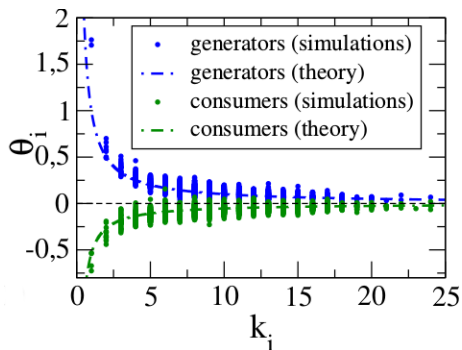
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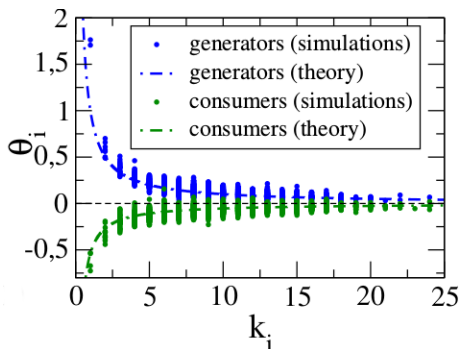
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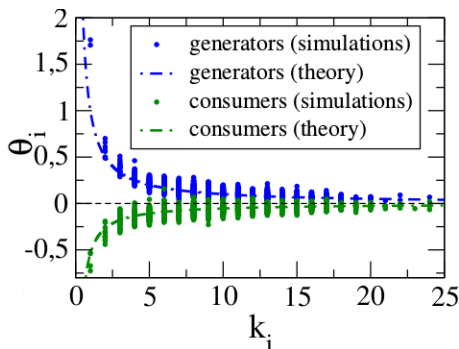
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- full Kuramoto dynamics live on low-dimensional subspace
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**Our route (Restrepo et al 2005, Sonnenschein et al 2014):**

- reducing complexity of Kuramoto dynamics through MF assumptions
- completely solve model



## More realistic MF model

$$\dot{\theta}_m = \omega_m + \lambda \sum_{n \in G} A_{mn} \sin(\theta_n - \theta_m) + \lambda \sum_{n \in C} A_{mn} \sin(\theta_n - \theta_m)$$

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With  $r_{mG} e^{i\psi_{mG}} \equiv \sum_{n \in G} A_{mn} e^{i\theta_n}$  and  $r_{mC} e^{i\psi_{mC}} \equiv \sum_{n \in C} A_{mn} e^{i\theta_n}$ ,  
perform old **Kuramoto trick**:

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### Approximation:

- (i)  $\psi_{mG} = \psi$  and  $\psi_{mC} = \psi_C \equiv 0$
- (ii)  $r_{mG} = x_m$  and  $r_{mC} = (k - x_m)$

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$$\dot{\theta}_{G,C}(k, x) = \omega_{G,C} + \lambda \{ x \sin[\psi - \theta_{G,C}(x)] - (k - x) \sin[\theta_{G,C}(x)] \}$$

$$\tan \psi = \frac{\sum_{x=0}^k x \binom{k}{x} g^x (1-g)^{k-x} \sin[\theta_G(x)]}{\sum_{x=0}^k x \binom{k}{x} g^x (1-g)^{k-x} \cos[\theta_G(x)]}$$

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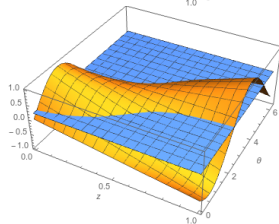
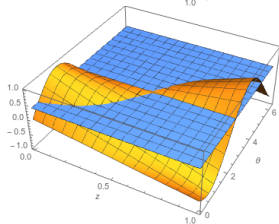
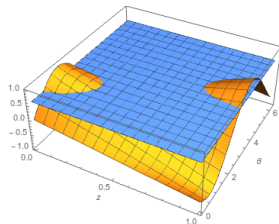
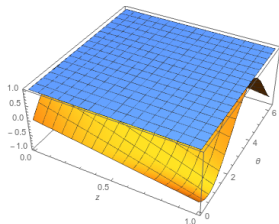
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**critical coupling:**

$$\lambda_{\psi} = [k \cos(\Psi/2)]^{-1} \text{ (even } k)$$

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$$\sin[\theta_{\Psi}(x)] = \frac{\omega_G/\lambda[k - (1 - \cos \Psi)x]}{k^2 - 2x(k - x)(1 - \cos \Psi)} + \frac{x \sin \Psi \sqrt{k^2 - 2x(k - x)(1 - \cos \Psi) - (\omega_G/\lambda)^2}}{k^2 - 2x(k - x)(1 - \cos \Psi)}$$

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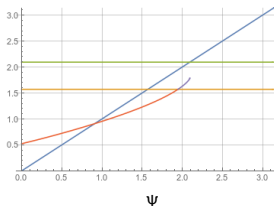
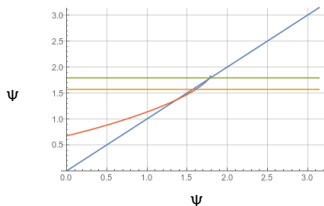
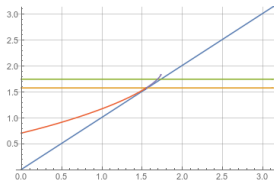
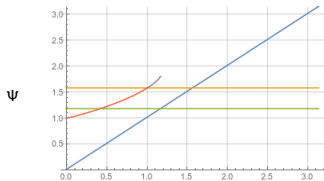
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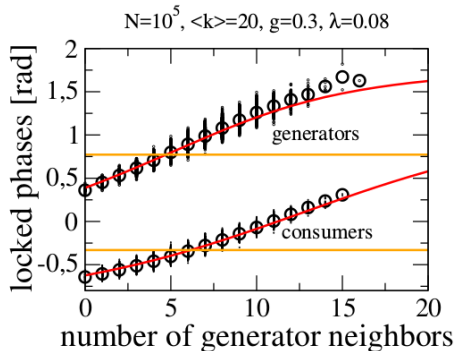
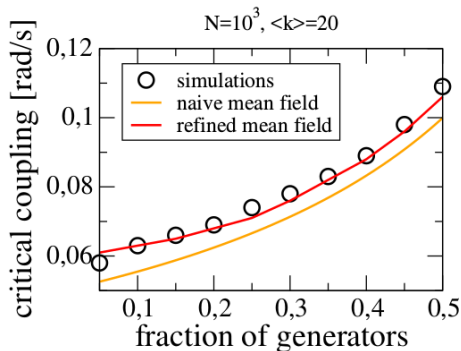
**constrain locked phases:**  $\theta_{\Psi}(x) \in [0, \pi)$  (no "splay states")

# Determine $\psi$ ...

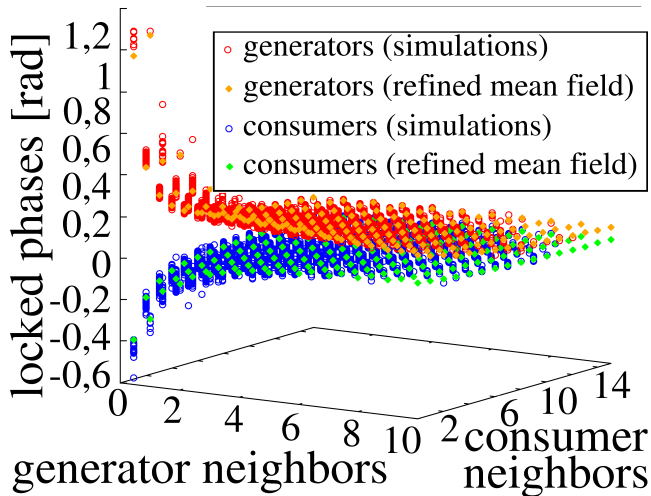
... by solving  $0 = \sum_{x=0}^k x \binom{k}{x} g^x (1-g)^{k-x} \sin[\theta_{-\psi}(k-x)]$



# Performance in random regular graphs



# Performance in random graphs



## Coexistence of stable solutions ...

(... with F. Suard, M. Vinyals and M. Velay):

$$\dot{\theta}_G(i) = \omega_G + \lambda \frac{i}{k G/N} \sum_{j=0}^k j \binom{k}{j} \left(\frac{G}{N}\right)^j \left(\frac{C}{N}\right)^{k-j} \sin[\theta_G(j) - \theta_G(i)]$$

$$+ \lambda \frac{k-i}{k G/N} \sum_{j=0}^k j \binom{k}{j} \left(\frac{G}{N}\right)^j \left(\frac{C}{N}\right)^{k-j} \sin[\theta_C(j) - \theta_G(i)]$$

$$\dot{\theta}_C(i) = -\frac{G}{C} \omega_G + \lambda \frac{k-i}{k C/N} \sum_{j=0}^k (k-j) \binom{k}{j} \left(\frac{G}{N}\right)^j \left(\frac{C}{N}\right)^{k-j} \sin[\theta_C(j) - \theta_C(i)]$$

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- set of quadratic equations in  $\sin[\theta_{G,C}(j)]$ ,  $\cos[\theta_{G,C}(j)]$
- solve with homotopy continuation methods (BERTINI), check stability
- no coexistence of stable equilibria for  $k \leq 7$  in random regular graphs



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- ... rules out splay states
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- ... is easily generalizable to other *configuration models*