

# Approximating the Conditional Density Given Large Observed Values via a Multivariate Extremes Framework, with Application to Environmental Data

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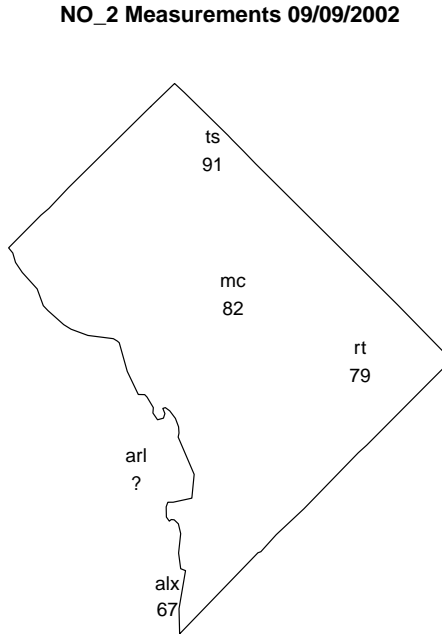
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## Abstract

Phenomena such as air pollution levels are of greatest interest when observations are large, but standard prediction methods are not specifically designed for large observations. We propose a method, rooted in extreme value theory, which approximates the conditional distribution of an unobserved component of a random vector given large observed values. Specifically, for  $\mathbf{Z} = (Z_1, \dots, Z_d)^T$  and  $\mathbf{Z}_{-d} = (Z_1, \dots, Z_{d-1})^T$ , the method approximates the conditional distribution of  $[Z_d | \mathbf{Z}_{-d} = \mathbf{z}_{-d}]$  when  $\|\mathbf{z}_{-d}\| > r_*$ . The approach is based on the assumption that  $\mathbf{Z}$  is a multivariate regularly varying random vector of dimension  $d$ . The conditional distribution approximation relies on knowledge of the angular measure of  $\mathbf{Z}$ , which provides explicit structure for dependence in the distribution's tail. As the method produces a predictive distribution rather than just a point predictor, one can answer any question posed about the quantity being predicted, and in particular one can assess how well the extreme behavior is represented.

Using a fitted model for the angular measure, we apply our method to nitrogen dioxide measurements in metropolitan Washington DC. We obtain a predictive distribution for the air pollutant at a location given the air pollutant's measurements at four nearby locations and given that the norm of the vector of the observed measurements is large.



**Figure 1:** Locations of the NO<sub>2</sub> monitors used in the Washington DC study. Locations are Alexandria (alx), McMillan (mc), River Terrace (rt), Takoma School (ts), and Arlington (arl). Also shown is the boundary of the District of Columbia. Measurements for September 9, 2002 are shown for the four locations we use for predicting the measurement at Arlington.

**Keywords:** Multivariate regular variation, Threshold exceedances, Angular or spectral measure, Air pollution, Nitrogen dioxide monitoring.

## 1 Introduction and Motivation

Nitrogen dioxide (NO<sub>2</sub>) is an air pollutant which is among those monitored by the US Environmental Protection Agency (EPA). Figure 1 shows NO<sub>2</sub> measurements at four locations in the Washington DC metropolitan area on September 9, 2002. This day's measurements are particularly large: each of the four measurements exceeds the 0.97 quantile of the empirical distribution for its location. Certainly, air pollution levels are of most interest when pollution levels are high. It is natural to ask, given the measurements at these four locations and given that they are large, what can be said about pollution levels at nearby unmonitored locations?

Linear prediction methods are questionable when the data are non-Gaussian, and a better approach may be to approximate the conditional density. Extreme value theory leads one to describe the joint tail with non-Gaussian distributions, and dependence in the tail is typically not well described by

covariances upon which linear prediction methods rely. In applications like the above air pollution example where interest lies in the large occurrences, approximating a conditional density allows one to answer important questions such as “Given that nearby locations’ measurements are large, what is the probability a certain unmonitored location exceeds some critical level?” or “Given that nearby locations’ measurements are large, what is a reasonable probabilistic upper bound for the air pollution level at an unmonitored location?”

Our method to approximate the conditional density is based on extreme value theory and is therefore specifically designed for instances when the observations are large. Extreme value theory provides a framework for describing the dependence found in the joint upper tail of the distribution, and at the same time, does not require knowledge of the entire joint distribution. In particular, we will assume that the joint distribution of the observations and the random variable we wish to predict are multivariate regularly varying, and we use the angular measure of this random vector to approximate the conditional density. By approximating the conditional density, we are able to address questions of the type posed above about the unobserved random variable.

In the next section, we provide some necessary background on extreme value theory. In Section 3 we discuss prediction for extremes; we review previous related work in Section 3.1 and then introduce our method in Section 3.2. In Section 4, we apply the prediction method to the Washington DC air pollution data. We conclude with a summary and discussion section.

## 2 Characterizing Extremes, Multivariate Regular Variation, and the Angular Measure

Extreme value analysis is the branch of statistics and probability theory whose aim is to describe the upper tail of a distribution. In this section we give a very brief overview of the discipline, particularly focusing on multivariate regular variation and the angular measure. There are a number of excellent resources if one wishes to delve more into the theory or practice of extreme value analysis. The book by de Haan and Ferreira (2006) gives a comprehensive overview of extreme value theory in the univariate, multivariate, and stochastic process settings. Beirlant et al. (2004) also gives a thorough treatment of the theory and gives a broad overview of recent statistical practice. Resnick (2007) focuses on the heavy-tailed case for both the univariate and multivariate settings. Coles (2001) gives an approachable introduction to statistical practice focusing primarily on maximum likelihood inference.

### 2.1 Extreme Value Analysis

Extreme value analysis is founded on asymptotic results that characterize a distribution’s upper tail by a limited class of functions. Statistical practice fits this class of functions to a subset of data which are considered extreme. Two approaches for choosing this subset of extreme data are commonly used: in the first, block (e.g., annual) maxima are extracted, in the second, observations which exceed a threshold are retained.

In the univariate case, asymptotic results lead one to model block maximum data with a generalized extreme value (GEV) distribution which consolidates the three extremal types (Fisher and Tippett, 1928; Gnedenko, 1943) into one parametric family. Threshold exceedance data are generally modeled via the generalized Pareto distribution (GPD) or an equivalent point process representation.

Statistical modeling of multivariate extremes is more complicated. Given a sequence of iid random vectors  $\mathbf{Y}_i = (Y_{i,1}, \dots, Y_{i,d})^T, i = 1, 2, \dots$ , classical multivariate extreme value theory considers the vector of renormalized element-wise maxima  $\frac{\mathbf{M}_n - \mathbf{c}_n}{\mathbf{b}_n}$  where the division is taken to be element-wise,  $\mathbf{M}_n = (\bigvee_{i=1}^n Y_{i,1}, \dots, \bigvee_{i=1}^n Y_{i,d})^T$ , and  $\bigvee$  denotes the maximum function. The theory shows the distribution of  $\frac{\mathbf{M}_n - \mathbf{c}_n}{\mathbf{b}_n}$  converges to a multivariate max-stable distribution (equivalently, multivariate extreme value distribution), which we characterize below. For threshold exceedance data, one must first define what it means for a random vector to exceed a threshold. Rootzen and Tajvidi (2006) define a multivariate GPD which is well-suited to describe threshold exceedances in which the threshold has been defined for each vector element. For the work below, we employ the framework of multivariate regular variation (Resnick, 2007) to describe threshold exceedances.

## 2.2 Multivariate Regular Variation

Multivariate regular variation is a notion that is used for modeling multivariate heavy-tailed data. This behavior is best seen via a natural decomposition into pseudo-polar coordinates. If non-negative  $\mathbf{Z} = (Z_1, \dots, Z_d)^T$  is a multivariate regularly varying random vector, then the radial component  $\|\mathbf{Z}\|$  decays like a power function; that is,  $\mathbb{P}(\|\mathbf{Z}\| > t) = L(t)t^{-\alpha}$ , where  $L(t)$  is a slowly varying function<sup>1</sup> at  $\infty$  and  $\alpha > 0$  is termed the tail index. The angular component,  $\|\mathbf{Z}\|^{-1}\mathbf{Z}$ , is described by a probability measure that lives on the unit sphere and which becomes independent of the radial component as the radial component drifts off to infinity. Central ideas in the more detailed treatment that follows are (1) the convergence to a measure  $\Lambda$  that serves as the intensity measure for a limiting point process, (2) that the limiting intensity measure is a product measure when described in terms of radial and angular components, and (3) the angular measure  $H$  which describes the distribution of the angular components.

There are several equivalent definitions of multivariate regular variation of a random vector. We say the non-negative random vector  $\mathbf{Z}$  is regularly varying if

$$\frac{\mathbb{P}(t^{-1}\mathbf{Z} \in \cdot)}{\mathbb{P}(\|\mathbf{Z}\| > t)} \xrightarrow{v} \Lambda(\cdot) \quad (1)$$

as  $t \rightarrow \infty$ , where  $v$  denotes vague convergence on  $\mathcal{C} = [0, \infty]^d \setminus \{\mathbf{0}\}$  and  $\|\cdot\|$  is any norm<sup>2</sup> (Resnick, 2007, Ch. 3). For any measurable set  $A \subset \mathcal{C}$  and scalar  $s > 0$ , the measure  $\Lambda$  has the scaling property

$$\Lambda(sA) = s^{-\alpha}\Lambda(A), \quad (2)$$

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<sup>1</sup> $L(t)$  is slowly varying if  $\lim_{t \rightarrow \infty} \frac{L(st)}{L(t)} = 1$ . Roughly,  $L(t)$  cannot go to zero or infinity faster than any power function.

<sup>2</sup>The compact sets in  $\mathcal{C}$  are all closed sets in  $[0, \infty]^d$ , that do not contain  $\mathbf{0}$ , i.e. the closed sets bounded away from  $\mathbf{0}$ .

from which one sees the power-function-like behavior. Choosing a sequence  $a_n$  such  $\mathbb{P}(\|\mathbf{Z}\| > a_n) \sim n^{-1}$ , one can obtain the sequential version of (1)<sup>3</sup>:

$$n\mathbb{P}\left(\frac{\mathbf{Z}}{a_n} \in \cdot\right) \xrightarrow{v} \Lambda(\cdot). \quad (3)$$

The transformation to polar coordinates  $R = \|\mathbf{Z}\|$  and  $\mathbf{W} = \|\mathbf{Z}\|^{-1}\mathbf{Z}$  naturally arises from the scaling property (2). If  $S_{d-1} = \{\mathbf{z} \in \mathcal{C} \mid \|\mathbf{z}\| = 1\}$  denotes the unit sphere under the chosen norm, then one can show there exists a probability measure  $H$  on  $S_{d-1}$  such that for any  $H$ -continuity Borel subset  $B$  of  $S_{d-1}$ ,

$$n\mathbb{P}\left(\frac{R}{a_n} > r, \mathbf{W} \in B\right) \longrightarrow r^{-\alpha} H(B). \quad (4)$$

Following Resnick (2007), we refer to  $H$  as the angular measure, although it is also referred to as the spectral measure. The advantage of the polar transformation is that the radial component  $R$  acts independently of the angular component  $\mathbf{W}$  whose behavior is captured by  $H$ .

From (4), one can characterize the tail behavior of  $\mathbf{Z}$  if one knows (or can estimate)  $\alpha$  and  $H$ . However, without further assumptions, this proves to be difficult as  $H$  can be any probability measure on  $S_{d-1}$ . For simplification purposes, in multivariate extremes it is often assumed that the components  $Z_j, j = 1, \dots, d$  of the random vector have a common marginal distribution, not just the common tail index that is implied by the general conditions of multivariate regular variation (e.g., Resnick (2002, Section 2)). There is no loss in generality by assuming specific margins (Resnick, 1987, Prop. 5.10). For the remainder, we will assume  $\mathbf{Z} = (Z_1, \dots, Z_d)^T$  is regularly varying with tail index  $\alpha = 1$  and that  $Z_j, j = 1, \dots, d$  have a common marginal distribution. Under this assumption, it follows that

$$\int_{S_{d-1}} w_1 dH(\mathbf{w}) = \int_{S_{d-1}} w_j dH(\mathbf{w}), \text{ for } j = 2, \dots, d, \quad (5)$$

providing some structure to the angular measure  $H$ . Furthermore, when  $\alpha = 1$  it is particularly useful to choose the  $L_1$  norm:  $\|\mathbf{z}\| = z_1 + \dots + z_d$ , for which the unit sphere is the simplex  $S_{d-1} = \{\mathbf{w} \in \mathcal{C} : w_1 + \dots + w_d = 1\}$ . With this norm,  $1 = \int_{S_{d-1}} dH(\mathbf{w}) = \int_{S_{d-1}} (w_1 + \dots + w_d) dH(\mathbf{w})$  and hence  $\int_{S_{d-1}} w_j dH(\mathbf{w}) = d^{-1}$ . In practice, the assumption of common marginals (or, for that matter, a common tail index) is rarely met. If the data that one intends to model arise from a  $d$ -dimensional random vector  $\mathbf{Y}$  for which the regularly varying  $\alpha = 1$  and common marginal assumptions do not hold, we presume there exist probability integral transforms  $T_j$  such that  $T_j(Y_j) = Z_j$  for  $j = 1, \dots, d$ . This preprocessing of the random variables is common in extreme value analyses (e.g., Cooley et al. (2010); Coles and Tawn (1991)) and can be viewed analogously to the preprocessing required to fit a stationary model to time series or spatial data.

One recognizes that (3) is the classic relationship characterizing convergence to a Poisson process, and it is often useful to think in terms of point processes when describing multivariate regular variation. From (3), the sequence of point processes  $N_n$  consisting of point masses located at  $\mathbf{Z}_i/a_n, i = 1, 2, \dots, n$  where  $\mathbf{Z}_i$  are iid copies of  $\mathbf{Z}$  converges to a nonhomogeneous Poisson process  $N$  with intensity measure  $\Lambda(\cdot)$  on  $\mathbb{B}(\mathcal{C})$  (Resnick, 2007, Section 6.2). We denote the corresponding intensity function by  $\Lambda(d\mathbf{z})$ , where  $\Lambda(A) = \int_A \Lambda(d\mathbf{z})$ . From (4), in terms of polar coordinates,

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<sup>3</sup>Other normalizing sequences are sometimes used, see Resnick (2007, p. 174)

$\Lambda(dr \times d\mathbf{w}) = r^{-2}drH(d\mathbf{w})$ . If the angular measure  $H$  is differentiable, then we refer to the angular density  $h(\mathbf{w})$ , and  $\Lambda(dr \times d\mathbf{w}) = r^{-2}h(\mathbf{w})drd\mathbf{w}$ .

Multivariate regular variation is a useful way to characterize the joint upper tail of a random variable  $\mathbf{Y}$  for a number of reasons. First, interest in extreme behavior is often greatest in cases when the tails are believed to be heavy (i.e., having asymptotic behavior like a power function), and multivariate regular variation provides a mathematical framework for such behavior. Even when the tails do not share a common tail index, marginal transformations as described above can be employed to utilize the framework. Second and more importantly, the angular measure  $H$  specifically describes the dependence found in the tails. Since our interest is in performing prediction when the observations are large, it is natural to use a framework specifically designed for describing tail dependence.

We perform prediction by approximating the conditional density. To do so, we will rely on a model for the angular measure  $H$ , and this model must be able to be evaluated for any  $\mathbf{w} \in S_{d-1}$ . There have been several parametric models proposed for  $H$  which meet the moment conditions (5). An early parametric model was the tilted Dirichlet model of Coles and Tawn (1991). Recently Cooley et al. (2010) and Ballani and Schlather (2011) employed a geometric approach to construct new parametric models. A semi-parametric model via a mixture of Dirichlet densities was introduced by Boldi and Davison (2007). Model fitting is done by Coles and Tawn (1991), Cooley et al. (2010), and Ballani and Schlather (2011) via a likelihood based on the point process representation for multivariate regular variation, while Boldi and Davison (2007) use both a Bayesian MCMC approach as well as an EM approach to fit their mixture model. Once fit, any of these models could be used for  $H$  in the prediction procedure we outline in Section 3.2.

There is further justification for using the framework of multivariate regular variation for modeling extreme values. The multivariate max-stable distributions obtained by the classical theory can be characterized by the angular measure  $H$ . If one assumes that the marginals of the limiting distribution are unit Fréchet ( $\mathbb{P}(Z \leq z) = \exp(-z^{-1})$ ); i.e., the domain of attraction of all regularly varying random variables with  $\alpha = 1$ ), then

$$\mathbb{P}\left(\frac{\mathbf{M}_n}{b_n} \leq \mathbf{z}\right) \xrightarrow{d} \exp \left[ -d \int_{S_{d-1}} \bigvee_{j=1}^d \left( \frac{w_j}{z_j} \right) dH(\mathbf{w}) \right]. \quad (6)$$

Here the normalizing sequence  $b_n = a_n/d$  to obtain the unit-Fréchet marginals. There have been parametric models developed which give closed-form expressions for subfamilies of multivariate max-stable distributions such as the asymmetric logistic (Tawn, 1990) and negative logistic (Joe, 1990), and these can be used to fit block maxima. Besides being max-stable, these distributions are multivariate regularly varying and we later use the logistic model (Gumbel, 1960) to simulate random vectors whose distribution function and limiting angular measure are both known in closed form.

### 3 Conditional Distribution Estimation and Prediction for Extremes

#### 3.1 Previous Work in Prediction for Extremes

There has been a small amount of work which has tried to devise methods for performing prediction for extremes. Davis and Resnick (1989, 1993) define a distance  $d$  between the components of a bivariate max-stable random variable, and suggest a method of prediction which minimizes the distance between the observed component and the predictor. Cragmille et al. (2006) offer a geostatistical approach to the problem of determining exceedances in a spatial setting by adjusting the loss function of the kriging predictor.

A recent important advance in the area of approximating a conditional distribution for extremes is the work of Wang and Stoev (2011), and we view the work in this paper as complementary. Wang and Stoev perform prediction for the case of max-stable random vectors. Let  $\mathbf{M}_n^{(d+p)} = (\mathbf{M}_n^{(d)}, \mathbf{M}_n^{(p)})^T$ , where  $\mathbf{M}_n^{(d)} = (M_{n,1}, \dots, M_{n,d})^T$ ,  $\mathbf{M}_n^{(p)} = (M_{n,1}, \dots, M_{n,p})^T$ , and where  $\mathbf{M}_n^{(d+p)}$  is assumed to be a max-stable random vector with a known distribution. Given data  $\mathbf{m}_n^{(d)} = (m_{n,1}, \dots, m_{n,d})^T$ , Wang and Stoev obtain approximate draws from  $\mathbf{M}_n^{(p)} \mid \mathbf{M}_n^{(d)} = \mathbf{m}_n^{(d)}$ . They accomplish this by sampling from spectrally-discrete max-stable models which can be represented as a max-linear combination of independent random variables. Using a spectrally discrete model would seem to be limiting, as it would imply that the corresponding angular measure would only have mass at discrete locations. However, it is known that any multivariate max-stable distribution can be approximated arbitrarily well by a max-linear model with a sufficient number of elements, and Wang and Stoev claim that their computational method can handle max-linear combinations on the order of thousands. Wang and Stoev (2011) apply their approach in the spatial setting and the results show the discrete approximation performs quite well.

The method we propose in the next section differs from that of Wang and Stoev (2011) in a number of important ways. Perhaps the most important difference is that, rather than performing prediction in a max-stable setting which would lend itself to data that are block maxima, our prediction method is best-suited for large observations; that is, the threshold exceedance case. Another difference is that, rather than successively drawing from the conditional distribution as Wang and Stoev do, we provide an analytic approximation to the conditional density given the observations are sufficiently large. Additionally, rather than relying on an approximation which corresponds to a discrete angular measure, our method instead relies on a parametric or semi-parametric model for the angular measure. Both methods involve non-trivial computation, although our method requires only the numerical computation of a one-dimensional integral, whereas Wang and Stoev's approach requires computation in fitting an adequate discrete approximation to the spectral measure and then in drawing from the conditional distribution.

#### 3.2 Approximating the Conditional Density when Observations are Large via the Angular Measure

Let  $\mathbf{Z}_{-d} = (Z_1, \dots, Z_{d-1})^T$ , and define  $\mathbf{z}_{-d}$  analogously. Working with the  $L_1$  norm, our goal is to approximate the distribution of  $[Z_d \mid \mathbf{Z}_{-d} = \mathbf{z}_{-d}]$  when  $\|\mathbf{z}_{-d}\| > r_*$  and  $r_*$  is large. Let us assume

that  $H$  is absolutely continuous with respect to Lebesgue measure on  $S_{d-1}$  and let  $h$  denote the corresponding density.

To approximate the conditional density, we employ the conditional pdf

$$f_{Z_d|Z_{-d}}(z_d | \mathbf{z}_{-d}) \approx \frac{\|\mathbf{z}\|^{-(d+1)} h\left(\frac{\mathbf{z}}{\|\mathbf{z}\|}\right)}{\int_0^\infty \|\mathbf{z}(t)\|^{-(d+1)} h\left(\frac{\mathbf{z}(t)}{\|\mathbf{z}(t)\|}\right) dt}, \quad (7)$$

where  $\mathbf{z} = (z_1, \dots, z_{d-1}, z_d)^T$  and  $\mathbf{z}(t) = (z_1, \dots, z_{d-1}, t)^T$ . This approximation arises from the point process representation for a regularly varying random vector as we show below. Consequently, the approximate conditional distribution utilizes the angular measure  $H$ , which characterizes the dependence in  $\mathbf{Z}$ 's components when  $\|\mathbf{Z}\|$  is large.

The first step in justifying the approximation (7) is to characterize the limiting measure  $\Lambda(\cdot)$  in terms of Cartesian rather than polar coordinates.

**Proposition 1** *Assume  $\mathbf{Z}$  is  $d$ -dimensional multivariate regularly varying with common marginal distributions, tail index  $\alpha = 1$ , and angular density  $h$ . Let  $N_n$  denote the sequence of point processes consisting of the point masses located at  $\{\mathbf{Z}_i/a_n, i = 1, 2, \dots, n\}$  where  $\mathbf{Z}_i$  are iid copies of  $\mathbf{Z}$ , and let  $N$  be the limiting point process as  $n \rightarrow \infty$ . Denote the intensity measure of  $N$  by  $\Lambda(\cdot)$ . Then  $\Lambda(d\mathbf{z}) = \|\mathbf{z}\|^{-(d+1)} h(\mathbf{z}/\|\mathbf{z}\|) d\mathbf{z}$ .*

**Proof:** The proof is a simple change-of-variables argument. Define the transformation  $p : (0, \infty) \times S_{d-1} \mapsto \mathcal{C}$  by  $\mathbf{z} := p(r, \mathbf{w}) = r\mathbf{w}$  and note that  $p$  is the inverse of the usual Cartesian-to-polar coordinate transform. To make the change of variables we need  $|\det J_{p^{-1}}|$ . It is known that  $|\det J_p| = r^{(d-1)}$  (Hogg et al. (2005), Example 3.37 (specific for the  $L_1$ -norm) and Song and Gupta (1997), Lemma 1.1 (for the general  $L_p$ -norm)). Thus  $|\det J_{p^{-1}}| = \|\mathbf{z}\|^{-(d-1)}$ .

Let  $A$  be an arbitrary set bounded away from  $\{\mathbf{0}\}$ , and consider  $\Lambda(A)$ .

$$\begin{aligned} \Lambda(A) &= \int_{(r, \mathbf{w}) \in p(A)} r^{-2} h(\mathbf{w}) dr \\ &= \int_{\mathbf{z} \in A} \|\mathbf{z}\|^{-2} h(\mathbf{z}/\|\mathbf{z}\|) \|\mathbf{z}\|^{-(d-1)} d\mathbf{z} \\ &= \int_{\mathbf{z} \in A} \|\mathbf{z}\|^{-(d+1)} h(\mathbf{z}/\|\mathbf{z}\|) d\mathbf{z} \end{aligned}$$

Thus  $\Lambda(d\mathbf{z}) = \|\mathbf{z}\|^{-(d+1)} h(\mathbf{z}/\|\mathbf{z}\|) d\mathbf{z}$ .  $\square$

**Remark:** The result is similar to Theorem 1 in Coles and Tawn (1991), which allows one to start with a known multivariate max-stable distribution with unit Fréchet marginals and find its corresponding angular measure. Coles and Tawn state “the drawback to the use of theorem 1 is that it can be applied only to MEVDs, of which very few have been obtained” [page 381]. It is important to note that our aim is somewhat the reverse of Coles and Tawn: we wish to start with



an angular measure, and obtain an approximation for the (conditional) density given the observed values are large.

Now, for any  $r_0 > 0$  and  $\mathbf{z} \in \mathbb{R}^d$  such that  $\|\mathbf{z}\| > r_0$ , let

$$F_{\mathbf{Z}/a_n}(\mathbf{z}, r_0) = \mathbb{P} \left( \frac{\mathbf{Z}}{a_n} \in [\mathbf{z}, \infty) \mid \frac{\|\mathbf{Z}\|}{a_n} > r_0 \right).$$

Then,

$$\begin{aligned} F_{\mathbf{Z}/a_n}(\mathbf{z}, r_0) &= \frac{\mathbb{P} \left( \frac{\mathbf{Z}}{a_n} \in [\mathbf{z}, \infty), \frac{\|\mathbf{Z}\|}{a_n} > r_0 \right)}{\mathbb{P} \left( \frac{\|\mathbf{Z}\|}{a_n} > r_0 \right)} \\ &= \frac{n \mathbb{P} \left( \frac{\mathbf{Z}}{a_n} \in [\mathbf{z}, \infty) \right)}{n \mathbb{P} \left( \frac{\|\mathbf{Z}\|}{a_n} > r_0 \right)} \\ &\rightarrow \frac{\Lambda([\mathbf{z}, \infty))}{\Lambda(\{\mathbf{z} \mid \|\mathbf{z}\| > r_0\})}, \text{ from (3)} \\ &= r_0 \Lambda([\mathbf{z}, \infty)), \text{ because } \int_{r>r_0} r^{-2} dr = r_0^{-1} \end{aligned} \tag{8}$$

$$= r_0 \int_{[\mathbf{z}, \infty)} \|\mathbf{z}\|^{-(d+1)} h(\mathbf{z} \|\mathbf{z}\|^{-1}) d\mathbf{z}, \text{ see Proposition 1.} \tag{9}$$

We wish to speak of  $f_{\mathbf{Z}/a_n}(\mathbf{z}, r_0)$ , a joint density of  $\mathbf{Z}/a_n$  given  $\|\mathbf{Z}\|/a_n > r_0$ . Heuristically from (9), we will assume that

$$f_{\mathbf{Z}/a_n}(\mathbf{z}, r_0) \rightarrow r_0 \|\mathbf{z}\|^{-(d+1)} h(\mathbf{z} \|\mathbf{z}\|^{-1}); \text{ for } \|\mathbf{z}\| > r_0 \tag{10}$$

as  $n \rightarrow \infty$ . More specifically, the convergence would be guaranteed if  $\frac{d}{d\mathbf{z}} F_{\mathbf{Z}/a_n}(\mathbf{z}, r_0)$  converged uniformly to  $r_0 \|\mathbf{z}\|^{-(d+1)} h(\mathbf{z} \|\mathbf{z}\|^{-1})$ , allowing us to switch the order of the limits associated with differentiation and as  $n \rightarrow \infty$ . See also Theorem 6.4 in Resnick (2007) in which regularly varying densities are described.

### Example: Bivariate Logistic Distribution

Let  $\mathbf{Z}$  have cdf  $\mathbb{P}(Z_1 \leq z_1, Z_2 \leq z_2) = \exp[-(z_1^{-1/\beta} + z_2^{-1/\beta})^\beta]$  for  $\beta \in (0, 1]$ .  $\mathbf{Z}$  is then said to have a bivariate logistic distribution, which is a known multivariate max-stable distribution, and which (more importantly for our purposes) is also regularly varying with common unit-Fréchet marginals  $\mathbb{P}(Z_j \leq z) = \exp(-z^{-1})$  for  $j = 1, 2$ . Coles and Tawn (1991) show that the angular density of the bivariate logistic is given by

$$h(\mathbf{w}) = \frac{1}{2} \left( \frac{1}{\beta} - 1 \right) (w_1 w_2)^{-1/\beta-1} \left( w_1^{-1/\beta} + w_2^{-1/\beta} \right)^{\beta-2}.$$

For a bivariate regularly varying random vector with unit Fréchet margins, it can be shown that

$a_n = 2n$  is a normalizing sequence such that  $\mathbb{P}(\|\mathbf{Z}\| > a_n) \sim n^{-1}$ . Now,

$$\begin{aligned}
\mathbb{P}\left(\frac{\mathbf{Z}}{2n} \in [\mathbf{z}, \infty)\right) &= \mathbb{P}(Z_1 > 2nz_1, Z_2 > 2nz_2) \\
&= 1 - \exp(-(2nz_1)^{-1}) - \exp(-(2nz_2)^{-1}) \\
&\quad + \exp\left[-\left((2nz_1)^{-1/\beta} + (2nz_2)^{-1/\beta}\right)^\beta\right] \\
&= (2nz_1)^{-1} + (2nz_2)^{-1} - \left((2nz_1)^{-1/\beta} + (2nz_2)^{-1/\beta}\right)^\beta + o(n^{-1})
\end{aligned} \tag{11}$$

and hence for  $\|\mathbf{z}\| > r_0$ ,

$$F_{\mathbf{Z}/2n}(\mathbf{z}, r_0) \rightarrow \frac{1}{2}r_0 \left( z_1^{-1} + z_2^{-1} - (z_1^{-1/\beta} + z_2^{-1/\beta})^\beta \right).$$

Differentiating this, we obtain the density

$$\begin{aligned}
f_{\mathbf{Z}/2n}(\mathbf{z}, r_0) &\rightarrow \frac{1}{2}r_0 (\beta^{-1} - 1) \left( z_1^{-1/\beta} + z_2^{-1/\beta} \right)^{\beta-2} z_1^{-1/\beta-1} z_2^{-1/\beta-1} \\
&= \frac{1}{2}r_0 (\beta^{-1} - 1) (z_1 + z_2)^{-3} \left( \left( \frac{z_1}{z_1 + z_2} \right)^{-1/\beta} + \left( \frac{z_2}{z_1 + z_2} \right)^{-1/\beta} \right)^{\beta-2} \\
&\quad \times \left( \frac{z_1}{z_1 + z_2} \right)^{-1/\beta-1} \left( \frac{z_2}{z_1 + z_2} \right)^{-1/\beta-1} \\
&= r_0 \|\mathbf{z}\|^{-3} h(\mathbf{z} \|\mathbf{z}\|^{-1}),
\end{aligned}$$

which agrees with (10). Similar arguments could be made for logistic models of dimension  $d > 2$ , but the inclusion/exclusion argument made in (11) becomes tedious.

Now, let us assume  $n$  is fixed, but large enough such that  $f_{\mathbf{Z}/a_n}(\mathbf{z}, r_0) \approx r_0 \|\mathbf{z}\|^{-(d+1)} h(\mathbf{z} \|\mathbf{z}\|^{-1})$ . We wish to approximate  $f_{\mathbf{Z}}(\mathbf{z}, r_*)$ , the density of  $\mathbf{Z}$  given that  $\|\mathbf{Z}\| > r_*$  where  $r_*$  is large. To obtain an approximation, we do a change-of-variables from  $\mathbf{Z}/a_n$  to  $\mathbf{Z}$ , which yields

$$\begin{aligned}
f_{\mathbf{Z}}(\mathbf{z}, r_*) &\approx r_0 \|\mathbf{z}/a_n\|^{-(d+1)} h(\mathbf{z} \|\mathbf{z}\|^{-1}) a_n^{-d} \\
&= r_* \|\mathbf{z}\|^{-(d+1)} h(\mathbf{z} \|\mathbf{z}\|^{-1}),
\end{aligned} \tag{12}$$

where  $r_* = a_n r_0$ , and thus is large.

Finally, consider the conditional distribution of  $[Z_d \mid \mathbf{Z}_{-d} = \mathbf{z}_{-d}]$  when  $\|\mathbf{z}_{-d}\| > r_*$  and  $r_*$  is large. Integrating to normalize the conditional density yields (7).

### 3.3 An Approximation Example

We investigate our approximation method via an example with a known distribution and angular measure. The trivariate logistic is a random vector with distribution  $\mathbb{P}(Z_1 \leq z_1, Z_2 \leq z_2, Z_3 \leq z_3) = \exp[-(z_1^{-1/\beta} + z_2^{-1/\beta} + z_3^{-1/\beta})^\beta]$  for  $\beta \in (0, 1]$ . The angular measure of the trivariate logistic is given by

$$h(\mathbf{w}) = \frac{1}{3} \left( \frac{1}{\beta} - 1 \right) \left( \frac{2}{\beta} - 1 \right) (w_1 w_2 w_3)^{-1/\beta-1} \left( w_1^{-1/\beta} + w_2^{-1/\beta} + w_3^{-1/\beta} \right)^{\beta-3}. \tag{13}$$

We first investigate the quality of the approximation as  $\|\mathbf{z}_{-d}\|$  increases and when  $\beta = 0.3$ . Since both the distribution and the angular measure are known, we can compare the approximated conditional density from (7) to the actual conditional density. The first three panels of Figure 2 show how the approximation improves as the magnitude of the observations grows. The top left panel shows that when the observed values are small ( $z_1 = 0.23, z_2 = 0.24$ ), the approximation to the true conditional density is poor. However, as the next two panels show, when the observations are sufficiently large, the approximation is quite good.

Next we use a simulation experiment to assess the skill in using (7) for approximating the conditional density when the conditioning observations are extreme. From the R package `evd` (Stephenson, 2002), we simulate 5000 trivariate logistic random vectors with  $\beta = 0.3$ . Let  $\mathbf{Z}_i = (Z_{i,1}, Z_{i,2}, Z_{i,3})^T$ ,  $i = 1, \dots, 5000$  denote the iid random variables and  $\mathbf{z}_i = (z_{i,1}, z_{i,2}, z_{i,3})^T$  be the realized values, of which only  $z_{i,1}$  and  $z_{i,2}$  are initially observed. We rank the realizations  $\mathbf{z}_i$  according to the sum of the observed values  $z_{i,1} + z_{i,2}$ . We then apply our approximation method to the largest 1000 of these simulations which corresponds to the condition  $z_{i,1} + z_{i,2} > 8.7$ . As each simulated random vector results in a unique conditional density approximation, we assess our method via a probability integral transform (PIT) or rank histogram (Gneiting et al. (2007), Wilks 2006, Sec. 7.7.2). Let  $f_{Z_{i,3}|Z_{i,1}, Z_{i,2}}(z_{i,3} | z_{i,1}, z_{i,2})$  be the approximated conditional density given by (7). On simulation  $i$ , if the observed values are large enough, we let  $p_i = \int_{-\infty}^{z_{i,3}} f_{Z_{i,3}|Z_{i,1}, Z_{i,2}}(s | z_{i,1}, z_{i,2}) ds$ , where  $z_{i,3}$  is the (previously unobserved) value of  $Z_{i,3}$ . We then construct a histogram for the values  $p_i$ . If the approximation is well-calibrated, then the PIT histogram should be flat, since there should be equal probability of  $p_i$  occurring in each bin. If the conditional density were correct, the counts in each bin would have a binomial ( $n = 1000, p = .1$ ) distribution and approximate quantiles for the sampling distribution can be generated under this null hypothesis. The bottom right panel indicates that the approximation seems to be quite good given that the observations are large, and given that the angular measure is known.

## 4 Application to Nitrogen Dioxide Air Pollution Measurements

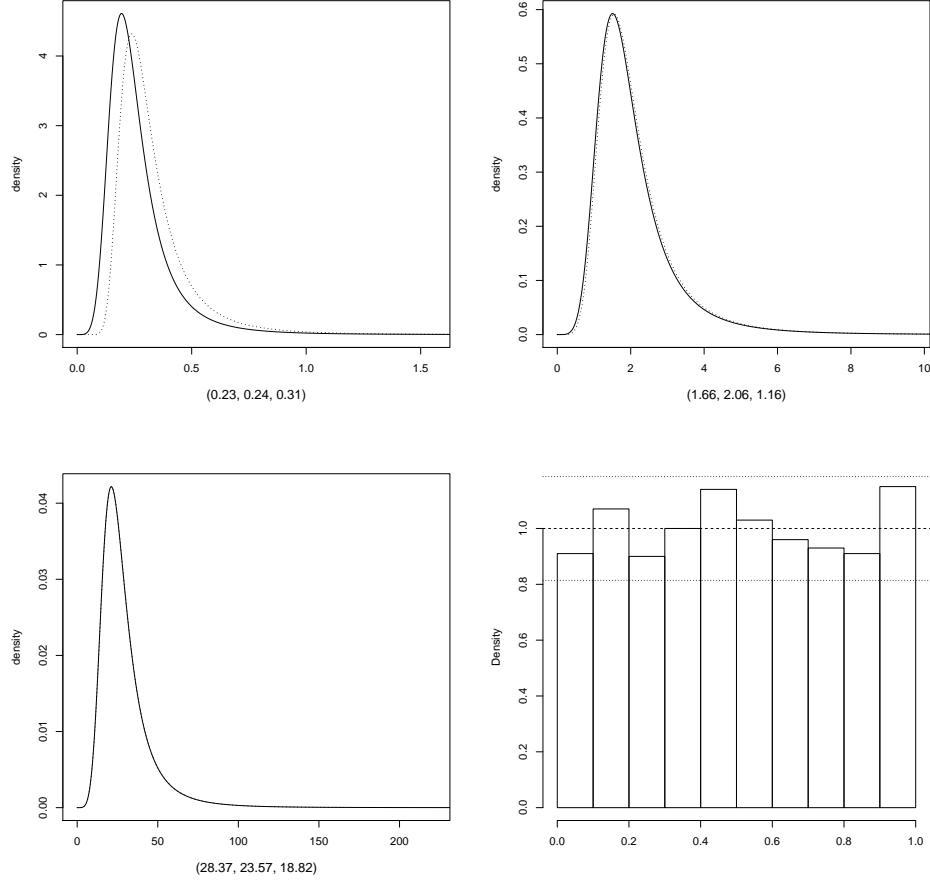
The nitrogen oxides ( $\text{NO}_x$ ) constitute one of the six common air pollutants for which the US EPA is required to set air quality standards by the Clean Air Act. Of the various nitrogen oxides, nitrogen dioxide ( $\text{NO}_2$ ) is the component of “greatest interest” and is used as an indicator of the entire group of  $\text{NO}_x$ <sup>4</sup>. According to the EPA fact sheet (EPA, 2010), short term  $\text{NO}_2$  exposures have been shown to cause adverse respiratory effects such as increased asthma symptoms. In January 2010, a new 1-hour  $\text{NO}_2$  standard was set at 100 parts per billion (ppb) to protect against adverse health effects due to short-term exposure to  $\text{NO}_2$ .

Under the guidelines set by the EPA’s Ambient Air Monitoring Program<sup>5</sup>, state and local agencies are charged with establishing and maintaining a network of air pollution monitoring stations. The EPA has made available data from these stations. Using an online tool<sup>6</sup>, we collected data from five stations located in Washington DC and nearby Virginia which were all active during the entire period from 1995-2010. The stations were Alexandria (site ID: 51-510-0009), McMillan (11-001-

<sup>4</sup><http://www.epa.gov/air/nitrogenoxides/>

<sup>5</sup><http://epa.gov/airquality/qa/monprog.html>

<sup>6</sup>[http://www.epa.gov/airexplorer/monitor\\_kml.htm](http://www.epa.gov/airexplorer/monitor_kml.htm)



**Figure 2:** Upper left, upper right, and lower left panels show three approximations of the conditional density of the third component of a trivariate logistic random variable given the first two components. The true conditional density is shown with the dotted line, the approximated density with the solid line. Note the different scales for the horizontal axis for these three figures. The approximation is poor when the observed values are small (upper left), but improves as these values become larger (upper right, lower left). The bottom right panel shows the PIT histogram of the largest 1000 (as determined by  $z_1 + z_2$ ) of 5000 total simulations. As the PIT histogram is flat, it shows that the approximation is good for these large observations. Dotted lines indicate the approximate 0.05 and 0.95 quantiles for the sampling distributions of each bin under the null hypothesis that the conditional distribution is correct.

0043), River Terrace (11-001-0041), Takoma School (11-001-0025), and Arlington (51-013-0020). The locations of these stations are shown in Figure 1.

Of course, air pollution measurements are of most interest when levels are believed to be high, and monitors only record pollutant levels at specific locations. We test our prediction method when observations are large on these five Washington-area stations. We aim to predict the  $\text{NO}_2$  level at the Arlington station, given the  $\text{NO}_2$  measurement at the other four stations. We choose  $\text{NO}_2$  because all five stations measured this pollutant, and  $\text{NO}_2$  appears to have the heaviest tail of the pollutants we examined.

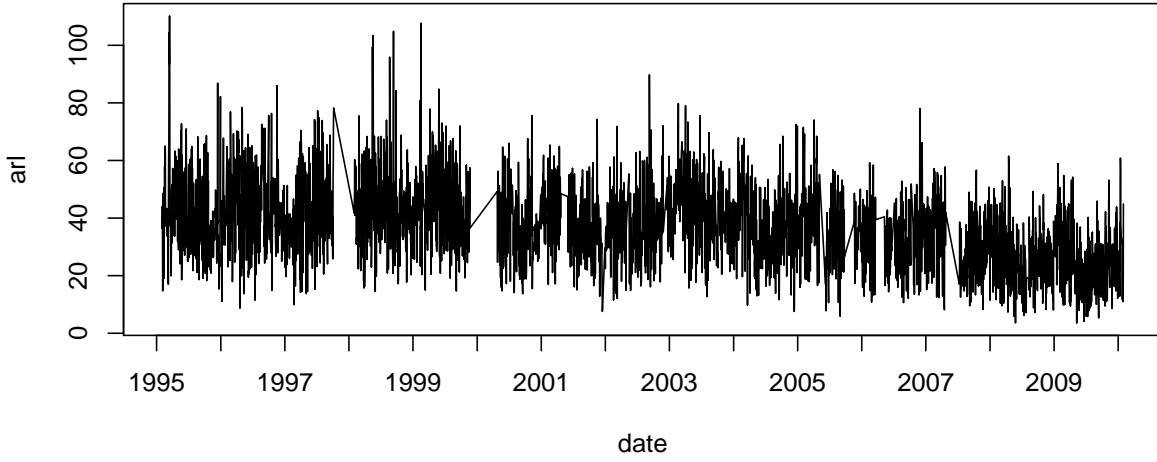
From each of the five stations, we extract the daily maximum  $\text{NO}_2$  measurement; all of the stations have over 5000 daily  $\text{NO}_2$  measurements recorded between Jan 1, 1995 - Jan 31, 2010 which meet EPA's daily summary quality requirements. From these, we keep only days in which all five stations have measurements, resulting in 4497 daily measurements. Finally, because the data are truncated to the nearest ppb, the empirical cdf appears quite discrete. Thus, we add a uniform random variable on the interval  $[-0.5, 0.5]$  to the data so that they behave more like the underlying continuous variable<sup>7</sup>.

Figure 3 shows the time series of the retained measurements at the Arlington station. Unlike other pollutants such as ground-level ozone, there does not appear to be a strong seasonal effect for  $\text{NO}_2$ . Although a very weak seasonal signal is detectable for a moving-average smoothed time series, this signal is hard to discern from a smoothed periodogram. It also appears that  $\text{NO}_2$  levels have decreased at this site over the study period, and the other stations show a similar, but weaker, trend. Checking for serial dependence in the data, we find the sample autocorrelation function of the deseasonalized data shows a highly significant correlation only at lag 1 ( $\hat{\rho}(1) = 0.35$ ). Figure 4 shows scatter plots of the measurements at the Arlington station versus the four stations used for prediction. The strong positive correlation between  $\text{NO}_2$  measurements shown in Figure 4 is indicative of that found among all pairs with the strongest sample correlation being 0.83 between Arlington and Alexandria and the weakest being 0.66 between Alexandria and McMillan. Figure 4 also shows that largest values can be coincident between stations. In our analysis that follows, we assume that the dependence in the upper tail of the joint distribution of  $\text{NO}_2$  measurements is not affected by the weak seasonality or the trend found in the data. We checked the trend assumption by fitting angular measure models to the first half and second half of the multivariate time series separately and found similar parameter estimates. We also ignore the serial dependence in the data, predicting the Arlington station's measurement using only the other four stations' measurements from that day.

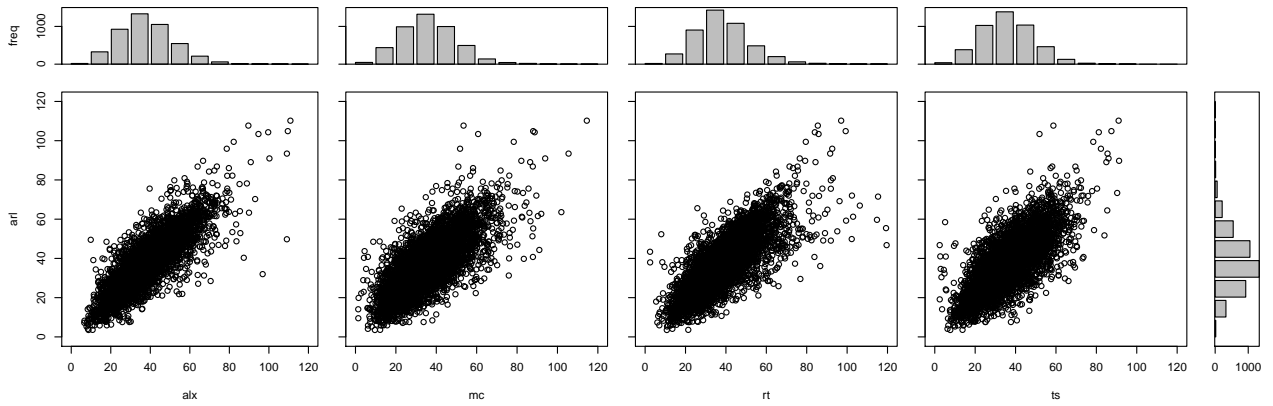
Let  $\mathbf{Y}_t = (Y_{t,1}, \dots, Y_{t,5})^T$  represent the random vector of measurements on day  $t$  at the five locations. Our first task is to estimate the angular measure which describes the tail dependence of the  $\text{NO}_2$  measurements at these locations. As formulated in Section 2.2, angular measure models assume a common marginal distribution with tail index  $\alpha = 1$ . To obtain common marginal distributions, we use the following procedure. Mean residual life plots (Coles, 2001, Sec 4.3.1) are used to select the 0.93 quantile as an appropriate threshold above which each location's data could be approximately Pareto-distributed. At each location, a generalized Pareto distribution (GPD) is fit to the data above the threshold via maximum likelihood. Letting  $\hat{F}_j$  be the estimated marginal distribution function formed by using the empirical distribution below the threshold and the fitted

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<sup>7</sup>Data available at <http://www.stat.colostate.edu/~cooleyd/DataAndCode/PredExtremes/nitrogenDioxideMsmts.RData>.



**Figure 3:** Time series plot of the retained measurements at the Arlington station.



**Figure 4:** Scatterplots of the measurements at the Arlington (arl) station verses the other four stations.

Site #	Location	$u$	$\hat{\psi}$	$\hat{\xi}$
1	alx	59.44	7.78(0.64)	0.07(0.06)
2	mc	56.80	8.29(0.70)	0.05(0.06)
3	rt	59.69	8.96(0.78)	0.10(0.07)
4	ts	55.51	6.67(0.55)	0.02(0.06)
5	arl	57.97	7.56(0.62)	0.07(0.06)

**Table 1:** Threshold and GPD estimates (and standard errors) of the tail for the five Washington DC area locations. The GPD is parametrized  $P(Y > y|Y > u) = \left(1 + \xi \frac{y-u}{\psi}\right)_+^{-1/\xi}$ . If  $\xi > 0$ , the tail index  $\alpha = 1/\xi$ .

GPD above (appropriately weighted by the observed exceedance probability), each location's data are transformed to have a unit Fréchet distribution:  $Z_{t,j} = (-\log(\hat{F}_j(Y_{t,j})))^{-1}$ . Table 1 summarizes the marginal tail estimates.

The data are divided into a training set and test set. The training set, consisting of two-thirds of the available data ( $n_{train} = 2998$ ), is used to fit a five-dimensional angular measure model. The test set, consisting of the other one-third ( $n_{test} = 1499$ ), is used to test the prediction method. Because of the decreasing trend, we construct the test set by extracting every third observation so that both the training and test sets would reflect the behavior over the entire study period.

The pairwise beta model (Cooley et al., 2010) is an angular measure model for dimension  $d > 2$  with parameters which help to control the amount of dependence between each pair of elements in the random vector. We fit the pairwise beta via maximum likelihood, and the likelihood arises by assuming that the point process relationship implied by (4) is exact for large observations (Coles and Tawn, 1991; Cooley et al., 2010). The largest observations were determined by  $\|z_t\|$ ; that is, the radial component of the transformed data, and the largest 210 observations (0.93 quantile) were used to fit the model.

The pairwise beta has angular density given by

$$\begin{aligned}
h(\mathbf{w}; \gamma, \boldsymbol{\beta}) &= K_d(\gamma) \sum_{1 \leq j < k \leq d} h_{j,k}(\mathbf{w}; \gamma, \beta_{j,k}), \text{ for } 0 < w_j < 1 \quad (14) \\
\text{where } h_{j,k}(\mathbf{w}; \gamma, \beta_{j,k}) &= (w_j + w_k)^{2\gamma-1} (1 - (w_j + w_k))^{\gamma(d-2)-d+2} \\
&\quad \times \frac{\Gamma(2\beta_{j,k})}{\Gamma^2(\beta_{j,k})} \left(\frac{w_j}{w_j + w_k}\right)^{\beta_{j,k}-1} \left(\frac{w_k}{w_j + w_k}\right)^{\beta_{j,k}-1}, \\
\text{and } K_d(\gamma) &= \frac{2(d-3)!}{d(d-1)\sqrt{d}} \frac{\Gamma(\gamma d + 1)}{\Gamma(2\gamma + 1)\Gamma(\gamma(d-2))},
\end{aligned}$$

is a normalizing constant. The estimated parameters for the fitted model are given in Table 2. In the pairwise beta model the magnitude of the  $\beta_{i,j}$  parameter is related to the level of dependence between the  $i$ th and  $j$  components; the fact that  $\hat{\beta}_{1,5}$  is the largest indicates that the Alexandria and Arlington stations show the strongest tail dependence.

For the test set, we assume that the Arlington station is not observed, and aim to approximate the conditional density of this station's NO<sub>2</sub> measurement given the measurements at the other four stations. Since our method is only valid when the observations are large, we perform prediction for

$\hat{\gamma}$	$\hat{\beta}_{1,2}$	$\hat{\beta}_{1,3}$	$\hat{\beta}_{1,4}$	$\hat{\beta}_{1,5}$	$\hat{\beta}_{2,3}$	$\hat{\beta}_{2,4}$	$\hat{\beta}_{2,5}$	$\hat{\beta}_{3,4}$	$\hat{\beta}_{3,5}$	$\hat{\beta}_{4,5}$
0.37	0.51	0.64	0.56	6.11	0.76	1.64	0.96	0.56	0.98	1.01
(0.03)	(0.18)	(0.28)	(0.19)	(2.59)	(0.44)	(1.08)	(0.51)	(0.20)	(0.51)	(0.61)

**Table 2:** Parameter estimates (and standard errors) for the pairwise beta angular measure model fit to the Washington D.C. NO<sub>2</sub> data.

the 105 test-set observations with the largest values of  $\|\mathbf{z}_{t,-5}\|$ . That is, we threshold at the empirical 0.93 quantile of the radial component (sum) of the transformed data at the observed locations. Using the fitted pairwise beta angular measure, the conditional density  $f_{Z_{t,5}|\mathbf{Z}_{t,-5}}(z_{t,5} | \mathbf{z}_{t,-5})$  was approximated using the procedure described in Section 3.2 for each of these top 105 observations<sup>8</sup>. The integration in the denominator of (7) was approximated using Simpson’s Rule. These were then back-transformed to obtain the conditional densities on the original scale  $g_{Y_{t,5}|\mathbf{Y}_{t,-5}}(y_{t,5} | \mathbf{y}_{t,-5})$ . Three of the approximated conditional densities can be found in the top row of Figure 5.

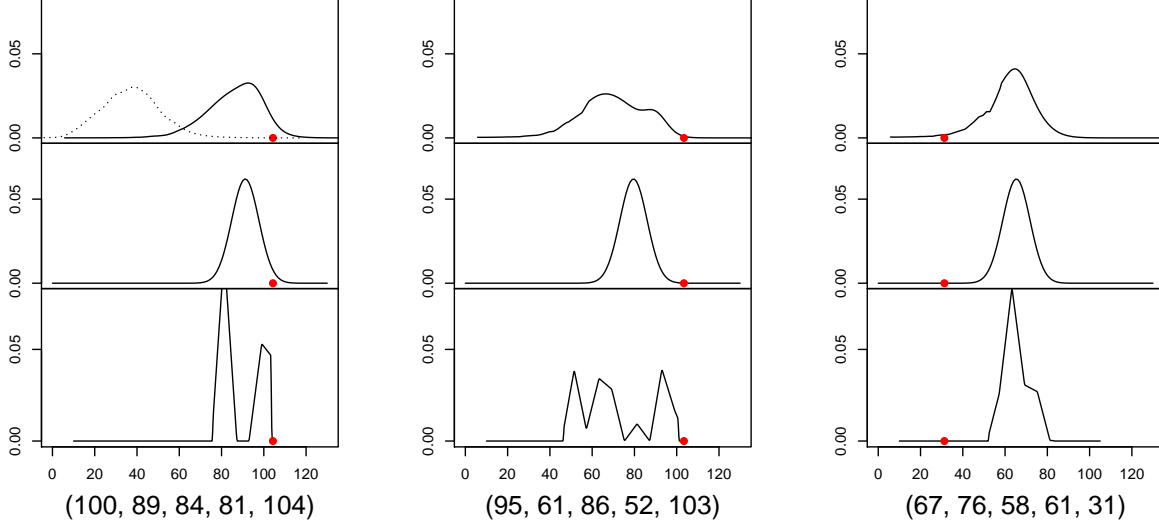
We compare our prediction method to two other approaches; best linear unbiased prediction (kriging) and indicator kriging (Cressie, 1993; Schabenberger and Gotway, 2005). Kriging is a prediction method that utilizes only mean and covariance information. At its most fundamental level, kriging does not make a distributional assumption, it provides a point prediction which corresponds to the best linear unbiased predictor in mean-square prediction error (MSPE), and additionally provides an estimate of the MSPE. To obtain confidence intervals, typically a Gaussian assumption is made. Furthermore, if one assumes the data arise from a Gaussian process, then the kriging estimate and MSPE correspond to the conditional expectation and variance. Since our method generates a conditional distribution, we will compare it to the conditional distribution provided by kriging under a Gaussian assumption.

Our kriging procedure is in parallel to the angular measure procedure above. The training set is used to formulate a model; here all 2298 observations are used to estimate the mean NO<sub>2</sub> levels at all five locations as well as to estimate the covariance matrix between the measurements at the locations. It is important to note that no spatial covariance function is fit, as our training set allows us to estimate the covariance matrix directly. Treating the mean and covariance as known, we then use simple kriging Cressie (1993) to obtain a point prediction at the Arlington location given the measurements at the other locations for the same 105 large observations in the test set. The MSPE is calculated from the estimated covariances, we use it to obtain the estimated conditional density at the Arlington location given the other measurements under a Gaussian assumption.

We also compare to indicator kriging (Cressie, 1993; Schabenberger and Gotway, 2005) which is a non-parametric version of kriging designed to provide estimates of  $\mathbb{P}(Y_d > u | Y_1, \dots, Y_{d-1})$  for a given threshold  $u$ . When performing indicator kriging, one first needs an estimate of the covariance matrix of the random variables corresponding to the indicators  $\mathbb{I}(Y_j > u)$  for all the  $j$  locations. At time  $t$ , given observations  $y_{t,1}, \dots, y_{t,d-1}$ , these are converted into indicators,  $\mathbb{I}(y_{t,j} > u)$  and ordinary kriging is used to estimate  $E[\mathbb{I}(Y_{t,d} > u) | \mathbb{I}(y_{t,1} > u), \dots, \mathbb{I}(y_{t,d-1} > u)] = \mathbb{P}(Y_{t,d} > u | \mathbb{I}(y_{t,1} > u), \dots, \mathbb{I}(y_{t,d-1} > u))$ . Repeating the analysis for various values of  $u$  allows one to estimate a conditional distribution, although there is no guarantee that the estimate will be monotonic.

<sup>8</sup>The code used to produce the results is available at <http://www.stat.colostate.edu/~cooleyd/DataAndCode/PredExtremes/PredExtremesFiles.zip>.



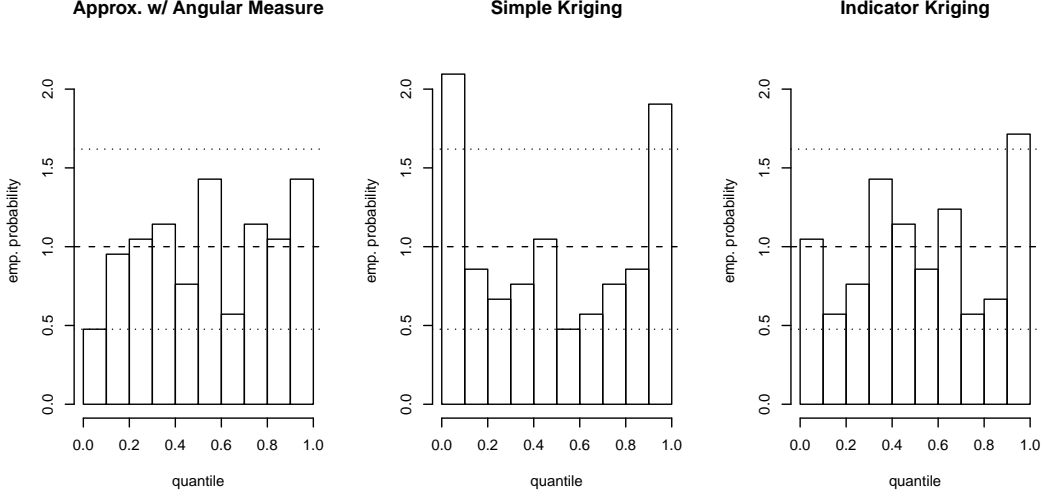


**Figure 5:** Comparison of the approximated conditional densities at the Arlington site given the measurements at the other sites:  $g_{Y_{t,5}|\mathbf{Y}_{t,-5}}(y_{t,5}|\mathbf{y}_{t,-5})$  for three different days with high measurements. Top figure in each column is the approximated conditional density via the angular measure, middle figure is from simple kriging, and the bottom figure is from indicator kriging. Below each figure is the vector of actual measurements at all five sites. The fifth element corresponds to the Arlington site which we are trying to predict and which is plotted with a dot in the figures. The dotted line in the upper left corresponds to the marginal distribution for the Arlington site.

Our indicator kriging analysis is again parallel to the angular measure and simple kriging analyses. We let  $u$  vary from 10-105 ppm with a step size of 0.25 ppm which covers the range of observations. The training set is used to estimate the covariance matrix of indicators at the various levels of  $u$ , and then indicator kriging is performed on each of the sets of observations in the test set. To guarantee that the conditional distribution is monotonic, we then perform a monotone quadratic smoothing spline regression (Meyer, 2008) on the estimates  $\mathbb{P}(Y_{t,d} > u \mid \mathbb{I}(y_{t,1} > u), \dots, \mathbb{I}(y_{t,d-1} > u))$  for all the values of  $u$ . Densities are obtained by differentiating the smoothing spline.

Conditional densities obtained by the angular measure method, simple kriging, and indicator kriging are shown in Figure 5 for three different days' data. In these three figures the conditional density approximated via the angular measure is less concentrated than the conditional density from simple kriging, and that proves to be the case in general. The angular measure can also be somewhat skewed or slightly bimodal depending on the combination of the observed measurements. Although indicator kriging is performed for each pollution level  $u$ , the conditional density as approximated by indicator kriging is very rough as there are only four locations.

We evaluate the performance of the three approaches using various methods. All comparisons are done at the original scale. To test the overall fit of the approximated conditional density, we again use the PIT histogram. Figure 6 shows the PIT histograms for all three methods. The PIT histogram for the angular measure method is relatively flat with perhaps some indication that the model is overestimating the probability in the lower tail resulting in too few observations falling in the first decile of the approximated conditional density. This could be due to the angular density model: the pairwise beta model fit to the data is certainly not the true model for the angular density



**Figure 6:** PIT histograms for the angular measure approach (left), simple kriging (center) and indicator kriging (right). Perfect estimation of the conditional density would be indicated by a flat histogram. Error bars are obtained for each decile from a binomial distribution ( $n = 105$ ,  $p = 0.10$ ).

which is unknown. It could also be due to threshold choice, although the parameter estimates of the pairwise beta model did not appear to be sensitive to the threshold. The kriging estimate with the Gaussian assumption shows a classic u-shape associated with underdispersion (Wilks, 2006, Sec 7.7.2). This model underestimates the probability of the observation occurring in the lower tail and also the upper tail. Indicator kriging also appears to underestimate the upper tail of the distribution resulting in has too many observations appearing in the highest decile of the approximated conditional distribution. Using the terminology of Gneiting and Raftery (2007), the PIT histograms indicate that the angular measure method is better (probabilistically) calibrated, particularly in the upper quantiles of the predictive distribution.

Another performance evaluation is to see how well each method estimates a quantile, and particularly, a high quantile. For instance, regulators might wish to have an accurate assessment of a high quantile of an unmonitored location given large observations nearby. Such an estimate could be used as a probabilistic upper bound, i.e., officials could state that they were 95% confident that the level at the unmonitored location was below a reported level. For each of the 105 large observations, we use the approximated conditional density from all three methods to estimate the 0.99, 0.95, 0.90, 0.75, and 0.50 quantiles. We examine coverage by calculating the proportion of actual observations that fell beneath these quantiles and also calculate each method's quantile verification score (QVS) (Gneiting and Raftery, 2007). Let  $g_{Y_{t,5}|\mathbf{Y}_{t,-5}}^{(m)}(s | \mathbf{y}_{t,-5})$  and  $G_{Y_{t,5}|\mathbf{Y}_{t,-5}}^{(m)}(s | \mathbf{y}_{t,-5})$  be the predictive density and cumulative distribution function for  $Y_{t,5} | \mathbf{Y}_{t,-5} = \mathbf{y}_{t,-5}$ , where  $m$  denotes the angular measure method ( $m = 1$ ), kriging ( $m = 2$ ), or indicator kriging ( $m = 3$ ). We have parameterized the QVS as in Friederichs and Hense (2007),

$$QVS^{(m)} = \sum_{t=1}^{105} \rho_{\tau}(y_{t,5} - q_{t,\tau}^{(m)}),$$

where  $q_{t,\tau}^{(m)} = G_{Y_{t,5}|\mathbf{Y}_{t,-5}}^{(m)\leftarrow}(\tau)$ , and  $\rho_{\tau}(u) = \tau u I(u \geq 0) + (\tau - 1)u I(u < 0)$ . A lower QVS score

Quantile	0.99		0.95		0.90		0.75		0.50	
	Cvg	QVS	Cvg	QVS	Cvg	QVS	Cvg	QVS	Cvg	QVS
Angular Measure	0.97	40.97	0.93	134.77	0.88	225.68	0.70	398.97	0.44	502.51
Simple Kriging	0.92	65.80	0.83	170.04	0.81	246.26	0.65	378.27	0.54	444.84
Indicator Kriging	0.90	67.80	0.86	153.41	0.83	238.63	0.73	377.20	0.49	452.68
Sampling Error	(0.01)	–	(0.02)	–	(0.03)	–	(0.04)	–	(0.05)	–

**Table 3:** Gives the skill of the different methods for assessing high quantiles. Coverage (Cvg) column reports the proportion of the observations at the Arlington location that fell beneath the quantile as calculated from the estimated conditional density and QVS column reports the quantile verification score (lower is better).

indicates better skill, but the scale of the QVS score depends on the quantile to which it is being applied. The QVS is a proper scoring rule, meaning that it is minimized if the predictive distribution corresponds to the ‘true’ distribution. Both the coverage and QVS results for the tested quantiles are shown in Table 3, as well as the sampling error assuming independent Bernoulli trials with  $p$  equal to the given quantile. The angular measure method does a superior job of estimating the high quantiles (0.99, 0.95, and 0.90) when the observations are large, whereas both simple kriging and indicator kriging underestimate these high quantiles. The angular measure method seems to be outperformed by indicator kriging for the 0.75 and 0.50 quantiles, although its coverage rates fall well within acceptable ranges when sampling error is accounted for.

Each method’s conditional density is essentially a probabilistic forecast, and scoring rules have been developed which provide an overall measure of the quality of probabilistic forecasts (Gneiting and Raftery, 2007). We assess the methods using two different proper scoring rules: the logarithmic score and the continuous rank probability score (CRPS). The logarithmic score for the prediction at time  $t$  is given by  $-\log(g_{Y_{t,5}|\mathbf{Y}_{t,-5}}^{(m)}(y_{t,5} | \mathbf{y}_{t,-5}))$  where  $y_{t,5}$  is the actual observation at the Arlington station and  $g_{Y_{t,5}|\mathbf{Y}_{t,-5}}^{(m)}$  is the estimated conditional density via the angular measure approach ( $m = 1$ ), kriging ( $m = 2$ ), and indicator kriging ( $m = 3$ ). The logarithmic score has an information-theoretic basis and corresponds to the Kullback-Leibler divergence between the predictive density  $g_{Y_{t,5}|\mathbf{Y}_{t,-5}}^{(m)}(s | \mathbf{y}_{t,-5})$  and the Kronecker delta function  $\delta_{s,y_{t,5}}$ . We assess the methods by the mean of the logarithmic scores

$$\frac{1}{105} \sum_{t=1}^{105} -\log(g_{Y_{t,5}|\mathbf{Y}_{t,-5}}^{(m)}(y_{t,5} | \mathbf{y}_{t,-5}))$$

over assumed independent realizations  $\mathbf{y}_t$ ,  $t = 1, \dots, 105$ . The mean logarithmic score is 3.93 for the angular measure approach, 4.24 for kriging, and infinity for indicator kriging, as 8 of the 105 of the observations  $y_{t,5}$  fall outside the support of the predictive distribution. Since a lower score is better, the angular measure method outperforms kriging and indicator kriging by this performance measure.

The logarithmic score has been criticized as it is not ‘robust’ to cases when observations fall outside the support of the distribution as in the indicator kriging case above. A popular alternative is the CRPS. For our example, the CRPS for a particular day  $t$  is given by

$$\int_{-\infty}^{\infty} \left( G_{Y_{t,5}|\mathbf{Y}_{t,-5}}^{(m)}(s | \mathbf{y}_{t,-5}) - \mathbb{I}\{s \geq y_{t,5}\} \right)^2 ds, \quad (15)$$

and can be understood as a non-linear function of the area between each method’s predictive cdf

$G_{Y_{t,5}|\mathbf{Y}_{t,-5}}^{(m)}(s | \mathbf{y}_{t,-5})$  and the heavyside function associated with the realized value  $y_{t,5}$ . The CRPS score rewards appropriate centering of the predictive distribution and narrowness of the predictive distribution otherwise known as ‘sharpness’. We assess the three methods by the mean of the CRPS scores for  $t = 1, \dots, 105$ . Given the PIT histograms and logarithmic scores, it is perhaps surprising that the mean CRPS scores associated with the angular measure method, kriging, and indicator kriging are 6.83, 6.36, and 6.21 respectively, indicating that by this performance measure, the angular measure method is performing worst. While all three methods produce predictive densities that are centered (i.e., the realized values exceed the predictive density’s median about half of the time), the predictive densities from kriging and indicator kriging are sharper than (but not as well calibrated as) the density produced by the angular measure method (Figure 5).

The CRPS score can be written as an integral with respect to a threshold  $s$  as in (15) or equivalently in terms the quantile function  $G_{Y_{t,5}|\mathbf{Y}_{t,-5}}^{(m)-1}(p)$  and integrated with respect to  $p \in (0, 1)$  (Gneiting and Ranjan, 2011). Further, the overall mean CRPS score can be decomposed into a mean CRPS score at each  $p$ , then integrated with respect to  $p$ . Gneiting and Ranjan (2011) suggest plotting the quantile score verses  $p$  as a diagnostic tool. When done for the three forecasts, the plot (not shown)<sup>9</sup> shows that the angular measure method outperforms the other methods for high quantiles, but both kriging and indicator kriging outperform the angular measure method for quantiles near 0.5, likely due to the increased sharpness of these methods. Since the quantile scores are naturally larger near values of 0.5, the overall mean CRPS scores for kriging and indicator kriging end up lower. Gneiting and Ranjan (2011) also discuss a quantile weighted CRPS score

$$\int_0^1 2 \left( \mathbb{I}\{y_{t,5} \leq G_{Y_{t,5}|\mathbf{Y}_{t,-5}}^{(m)-1}(p)\} - p \right) \left( G_{Y_{t,5}|\mathbf{Y}_{t,-5}}^{(m)-1}(p) - y_{t,5} \right) v(p) dp$$

where one can choose the weight function  $v(q)$  to emphasize quantiles of interest. Letting  $v(q) = \mathbb{I}\{v(q) > 0.85\}$ , the mean weighted CRPS scores for the angular measure method, kriging, and indicator kriging are 0.50, 0.57, and 0.55.

## 5 Summary and Discussion

In this work, we obtain an approximation for the distribution of a component of a regularly varying random vector given that the observed components are large. We apply the approximation to estimate the conditional distribution of an air pollutant given nearby measurements that are large. Results show that our method outperforms traditional spatial prediction methods at capturing the conditional distribution of the random variable when the observations are large. PIT histograms show that our method is better calibrated, and the method proves to be much better suited for obtaining probabilistic upper bounds of the pollutant level. For example, the estimated 95% quantile provided by kriging and indicator kriging were too low and the actual exceedance rates were 17% and 14% respectively. The exceedance rate of the angular measure method’s estimated 95% quantile was 7% and was within sampling error of 5%.

We believe that this is the first work to perform prediction using extremes techniques in a threshold exceedance setting. The classic theory that leads to max-stable distributions and processes is quite

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<sup>9</sup><http://www.stat.colostate.edu/~cooleyd/Papers/PredExtremes/quantileScorePlot.pdf>

elegant and forms the foundation for all of extreme value theory. Statistical practice utilizing multivariate max-stability generally requires one to obtain component-wise block maximum data, and such data can be viewed as “artificial” in the sense that one models data vectors that are likely to have never occurred, since the block maxima are likely to occur at different times. It seems natural to try and attempt to describe large concurrent observations, and the framework of multivariate regular variation allows this.

Our method relies on an adequate angular measure model. There has been some renewed interest of late in constructing flexible models which meet the moment conditions (5) (Cooley et al., 2010; Ballani and Schlather, 2011; Boldi and Davison, 2007). However, no model with a finite parameterization can completely describe the possible angular measures, and the existing models may not prove to adequately model every multivariate data set. These models become unwieldy as the dimension increases beyond moderate levels ( $d \approx 5$ ). Certainly there remains a need for flexible multivariate extremes models.

Although we apply our method to multivariate time series data, we do not make use of any temporal dependence in the data. Our method proceeds as if the sequence of multivariate random vectors are iid. One could extend the method by allowing the marginal distributions to vary in time; such extreme value models are regularly used (e.g., Beirlant et al. (2004, ch. 7)) and might be required if, for instance, the seasonality of this data had been more influential.

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