The Elo rating system in chess

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Saclay
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Outline

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Examples of ratings/rankings

Women's Volleyball

1. China	330
2. USA	298
3. Serbia	272
4. Brazil	230
5. Russia	206

Men's Tennis

1. Nadal	9465
2. Federer	7505
3. Murray	6790
4. Zverev	4470
5. Čilić	4155

Chess

1. Carlsen	2827
2. Vachier-Lagrave	2804
3. Kramnik	2803
4. Aronian	2802
5. Caruana	2799

Coding

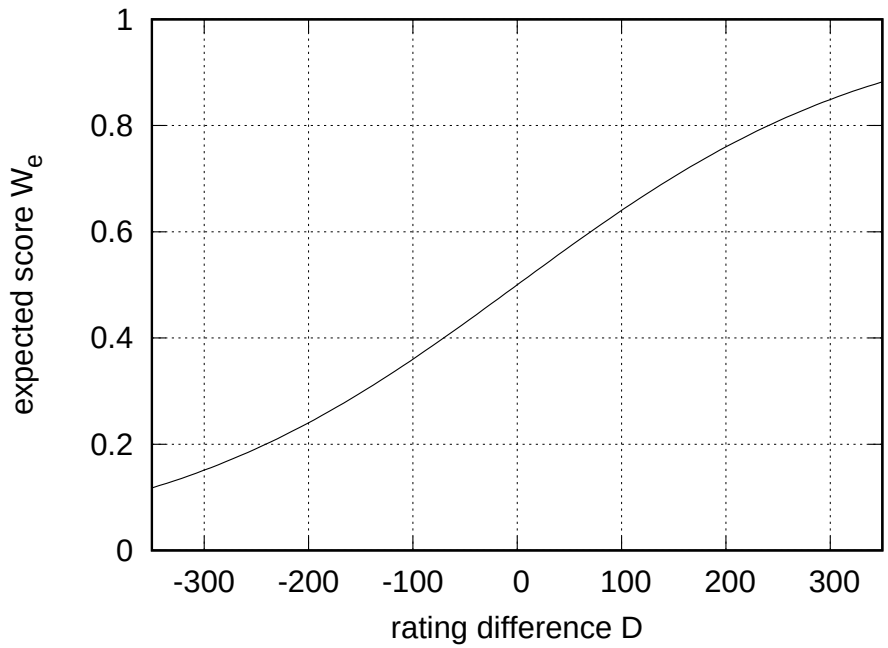
1. Petr	3676
2. rng_58	3670
3. tourist	3427
4. Um_nik	3274
5. xudyh	3271

Rating update equation

$$X_{n+\ell} = X_n + K(W_a - W_e)$$

X_n	rating after n games
K	scale factor
W_a	actual score
W_e	expected score

$$W_e(D) = \frac{1}{1 + 10^{-D/400}}$$



Rating update: example calculation

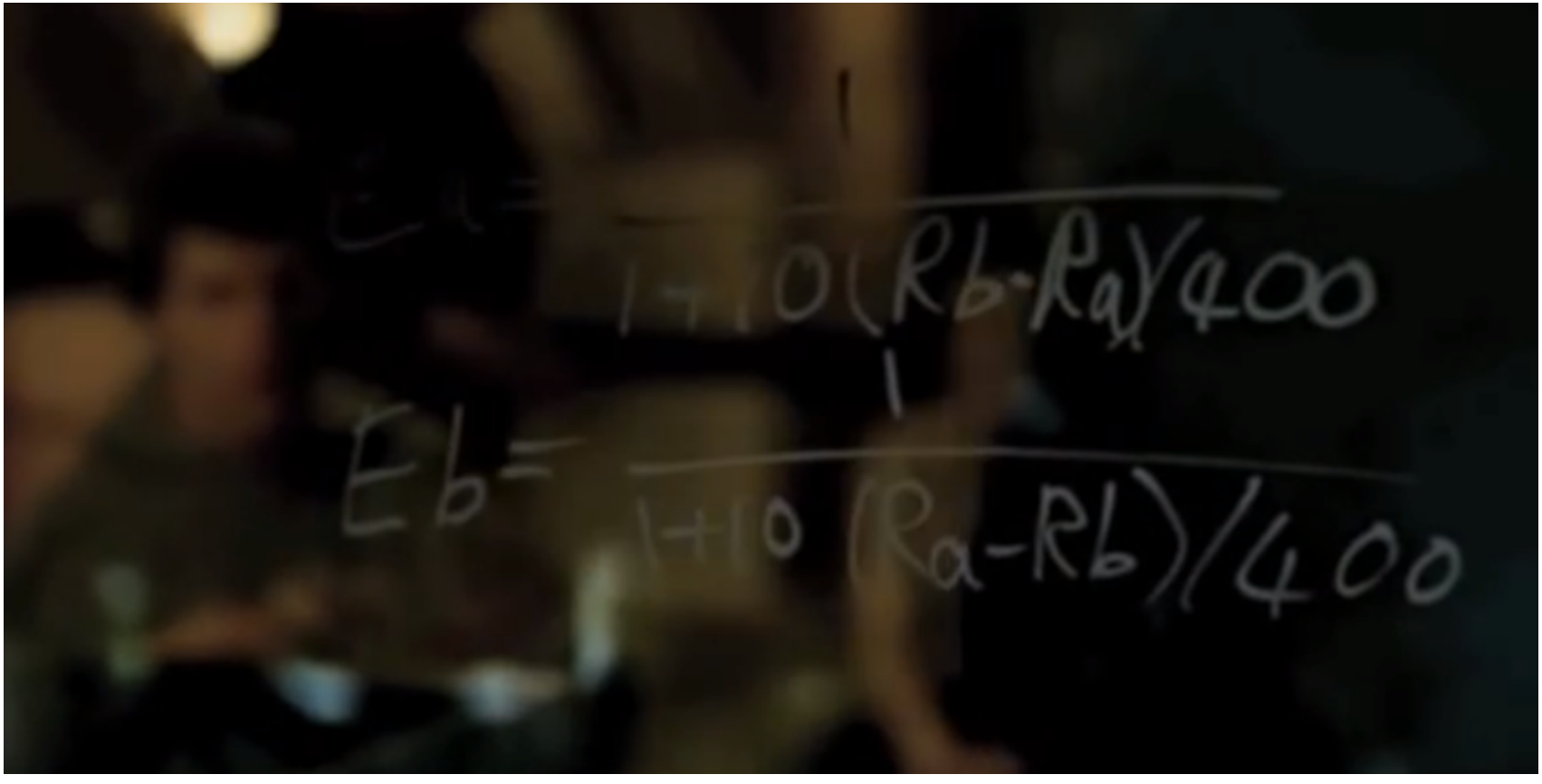
opponent's rating	result
2515	½
2472	1
2430	1
2608	0

$$X_n = 2500$$
$$K = 15$$
$$W_a = 2.5$$
$$W_e = 1.97$$

New rating:

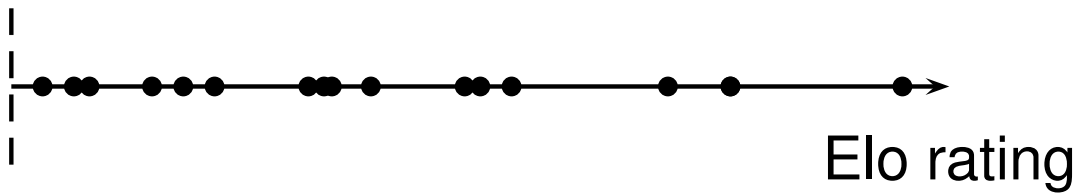
$$\begin{aligned} X_{n+4} &= X_n + K(W_a - W_e) \\ &= 2500 + 7.95 \\ &= 2508 \qquad \text{(after rounding)} \end{aligned}$$

Incorrect formula in *The Social Network*



Incorrect chessboard in watercolor

Complicated non-stationary dynamics



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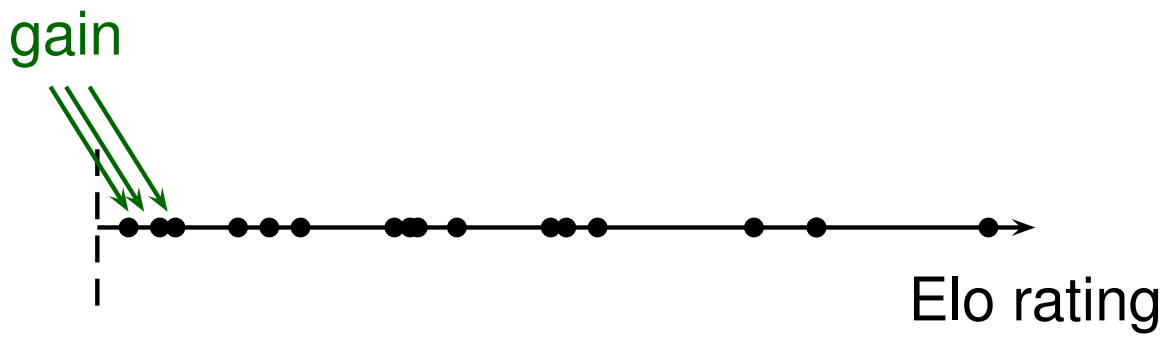

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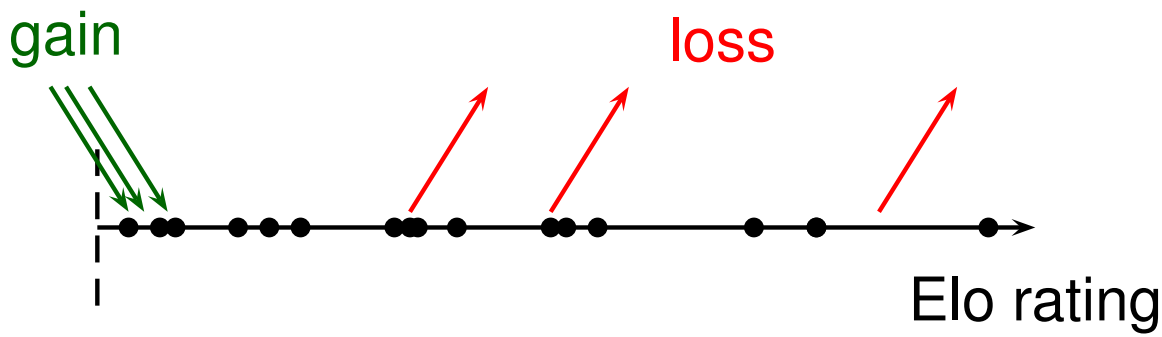

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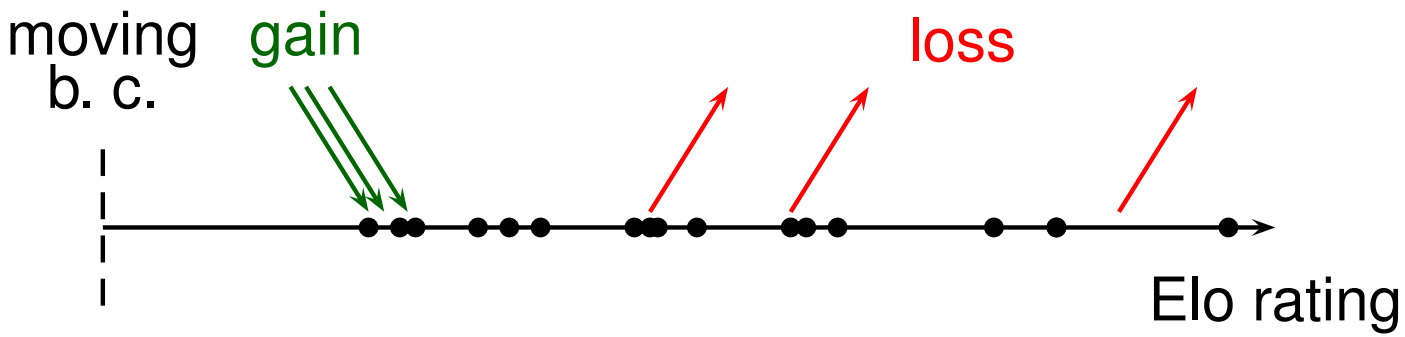

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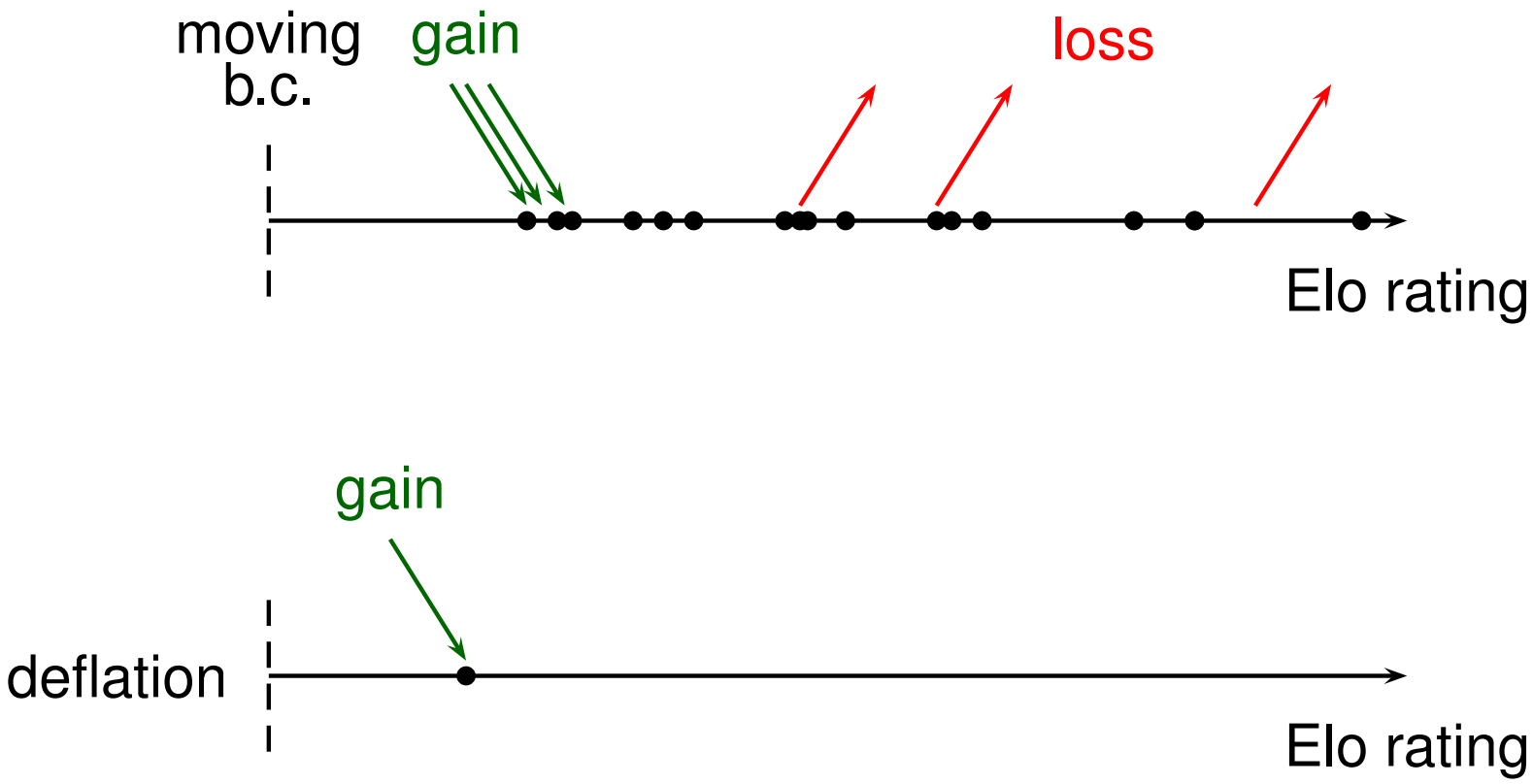

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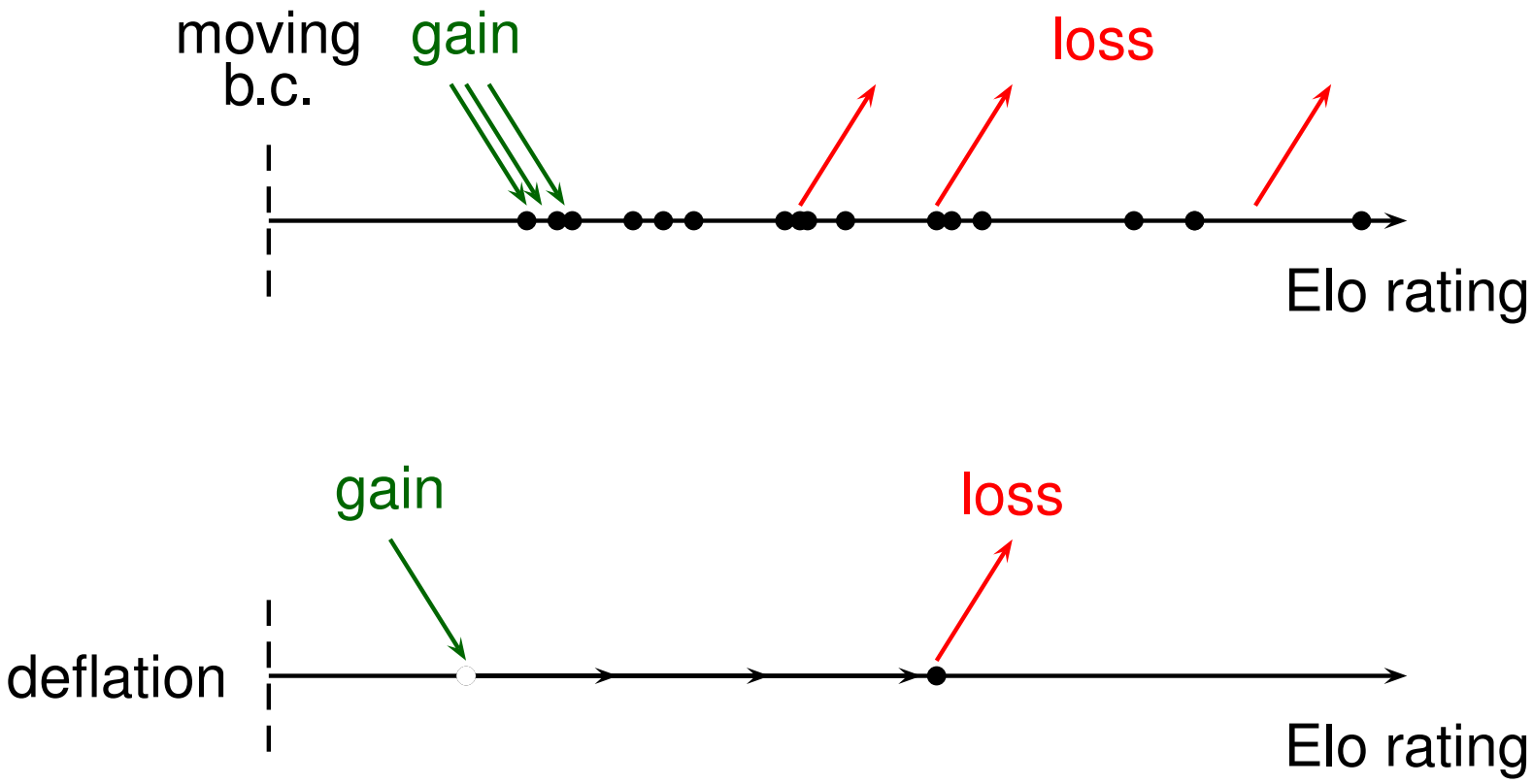

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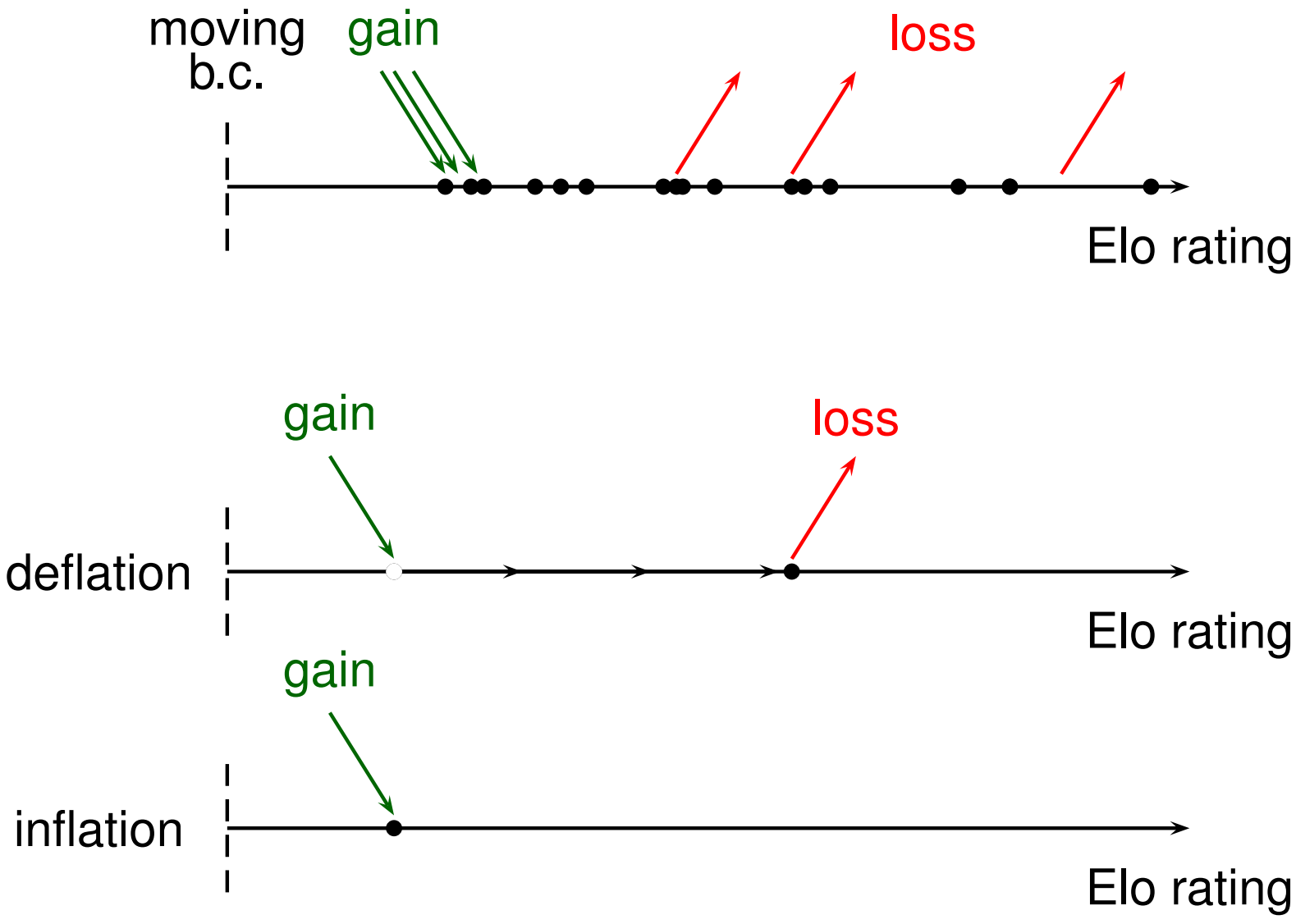

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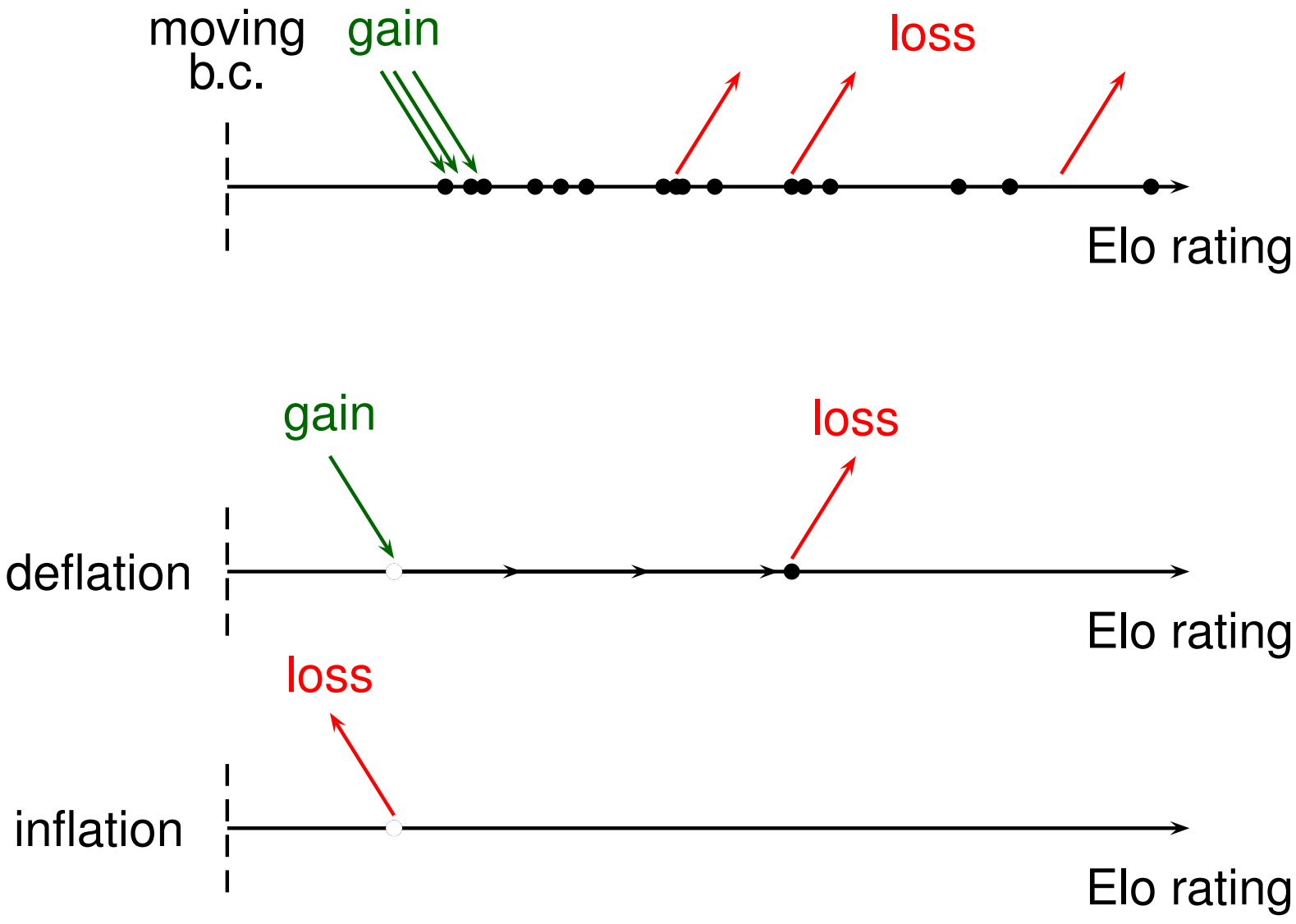
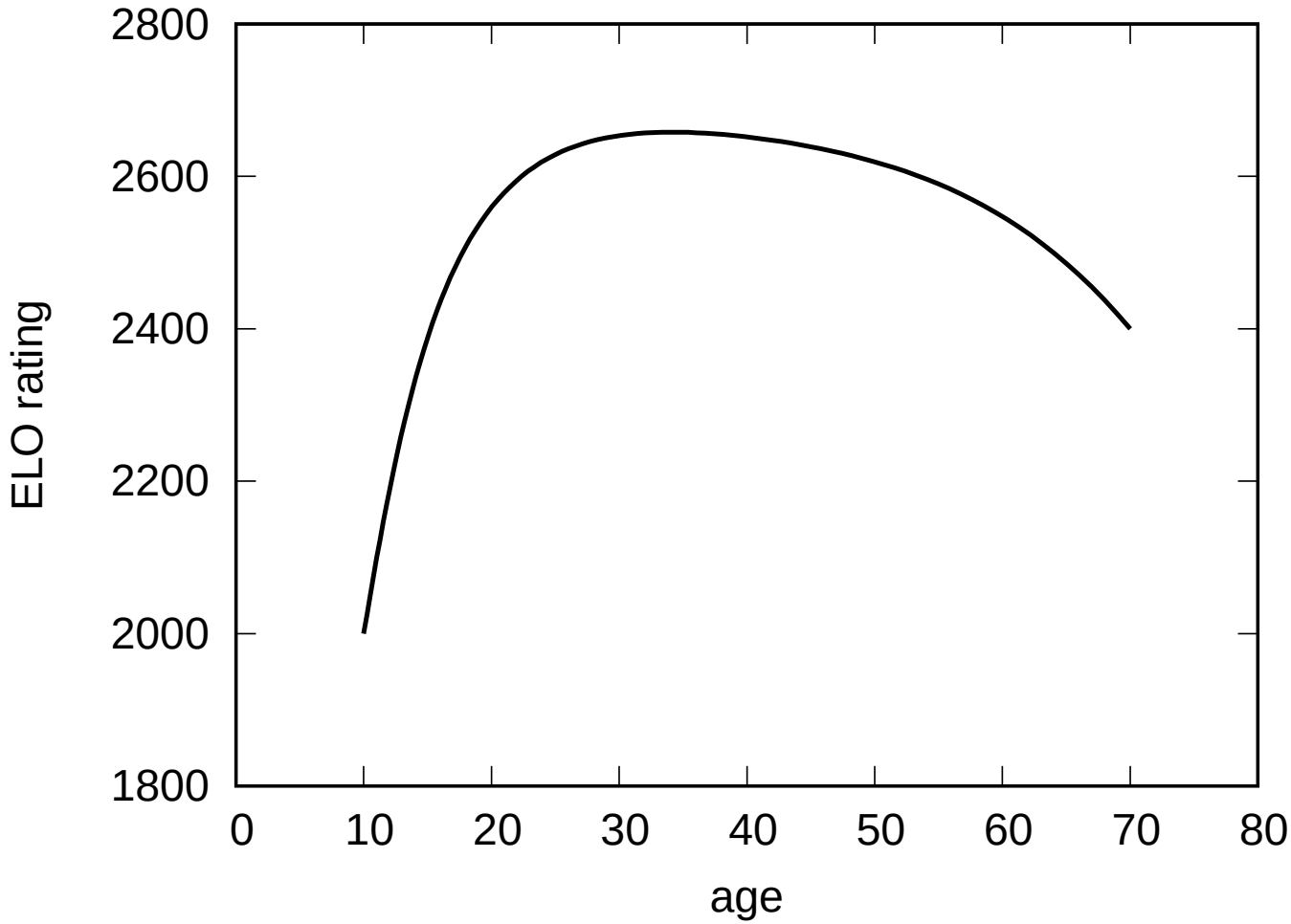


Outline

Lifetime evolution of a player's rating



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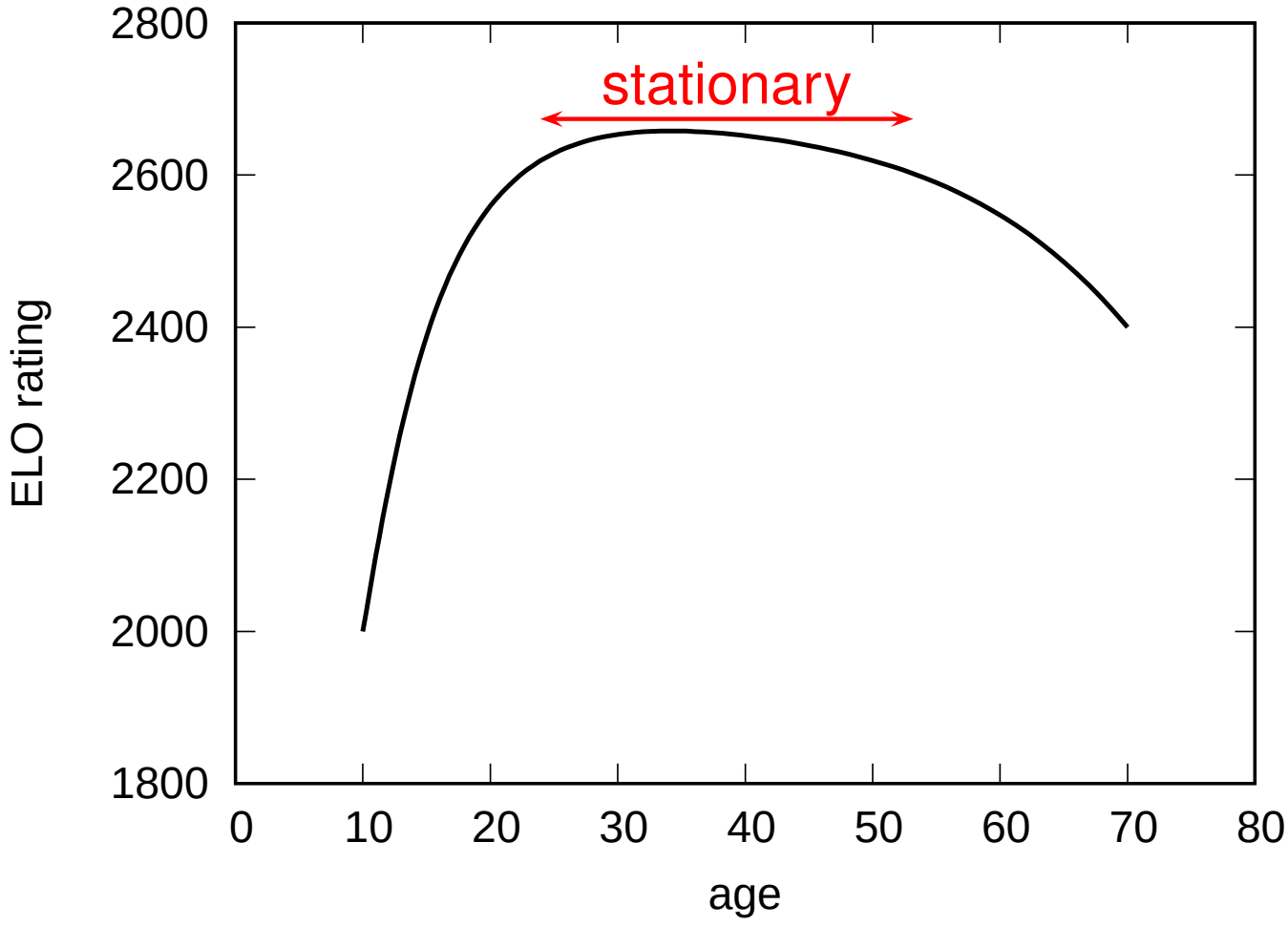

AR(1) modeling

$X_{n+\ell} = X_n + K(W_a - W_e)$

Key assumptions:

- player's rating fluctuates around an intrinsic value $X^\star$.
- average rating of opponents $\bar{X}$ is close to $X^\star$.

As a consequence:

$$\begin{aligned} W_a &\approx W_e(X^\star - \bar{X}, \ell) \\ &\approx \ell W_e(0) - \ell \bar{X} W_e'(0) + W_e'(0) \sum_{i=1}^{\ell} X_i^\star \\ &\approx \ell W_e(0) - \ell \bar{X} W_e'(0) + \ell W_e'(0) X^\star + \sqrt{\ell} W_e'(0) \sigma Z, \end{aligned}$$

where approximately

$$Z \sim N(0, 1).$$

AR(1) modeling

$X_{n+\ell} = X_n + K(W_a - W_e)$

$$X_{n+\ell} = X_n + K[W_a(X^\star - \bar{X}, \ell) - W_e(X_n - \bar{X}, \ell)]$$

AR(1) modeling

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\begin{aligned}
X_{n+\ell} &= X_n + K[W_a(X^\star - \bar{X}, \ell) - W_e(X_n - \bar{X}, \ell)] \\
&\approx [1 - \ell K W_e'(0)]X_n + \ell K W_e'(0)X^\star + \sqrt{\ell} K W_e'(0)\sigma Z_{n+\ell}
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This is an AR(1) process with

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\phi &= 1 - \ell K W_e'(0) \\
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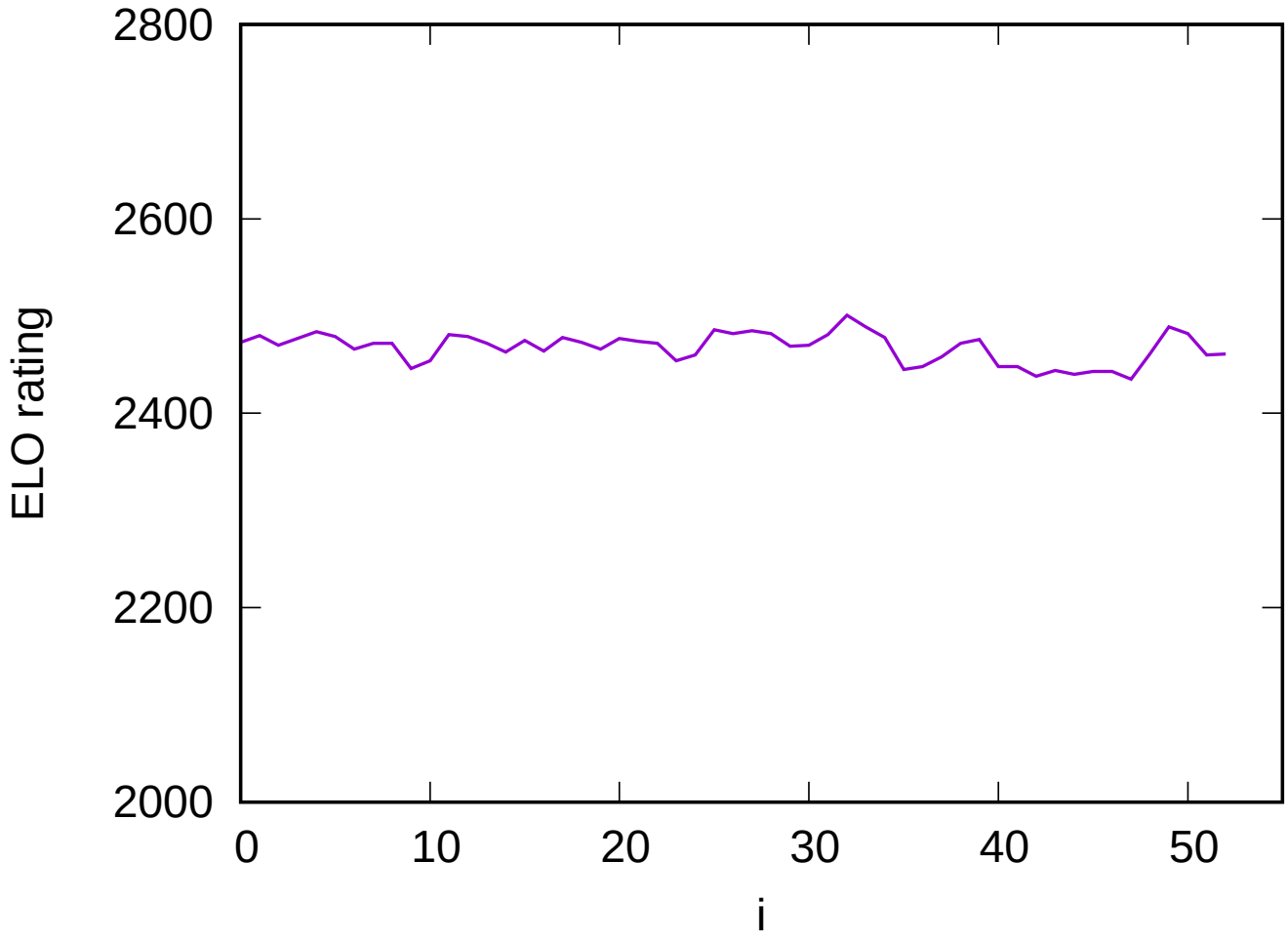
Putting in some realistic numbers:

$$
\begin{aligned}
W_e'(0) &\approx 1/700 \\
K &\approx 10 \\
\ell &\approx 10\text{-}20 \\
\implies \phi &\approx 0.7\text{-}0.85
\end{aligned}
$$

Only $X^\star$ and σ to fit!

Outline

AR(1) analysis



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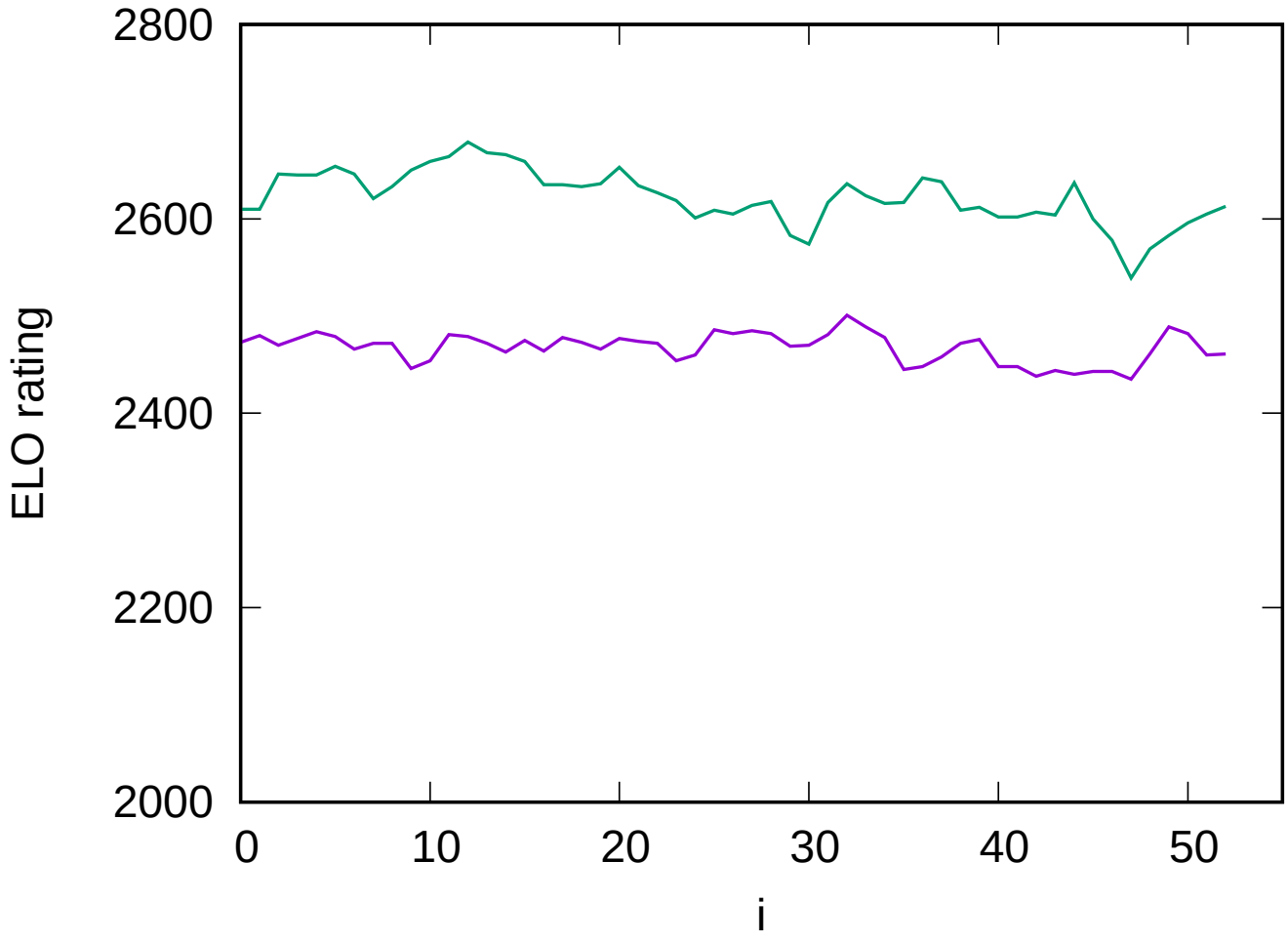

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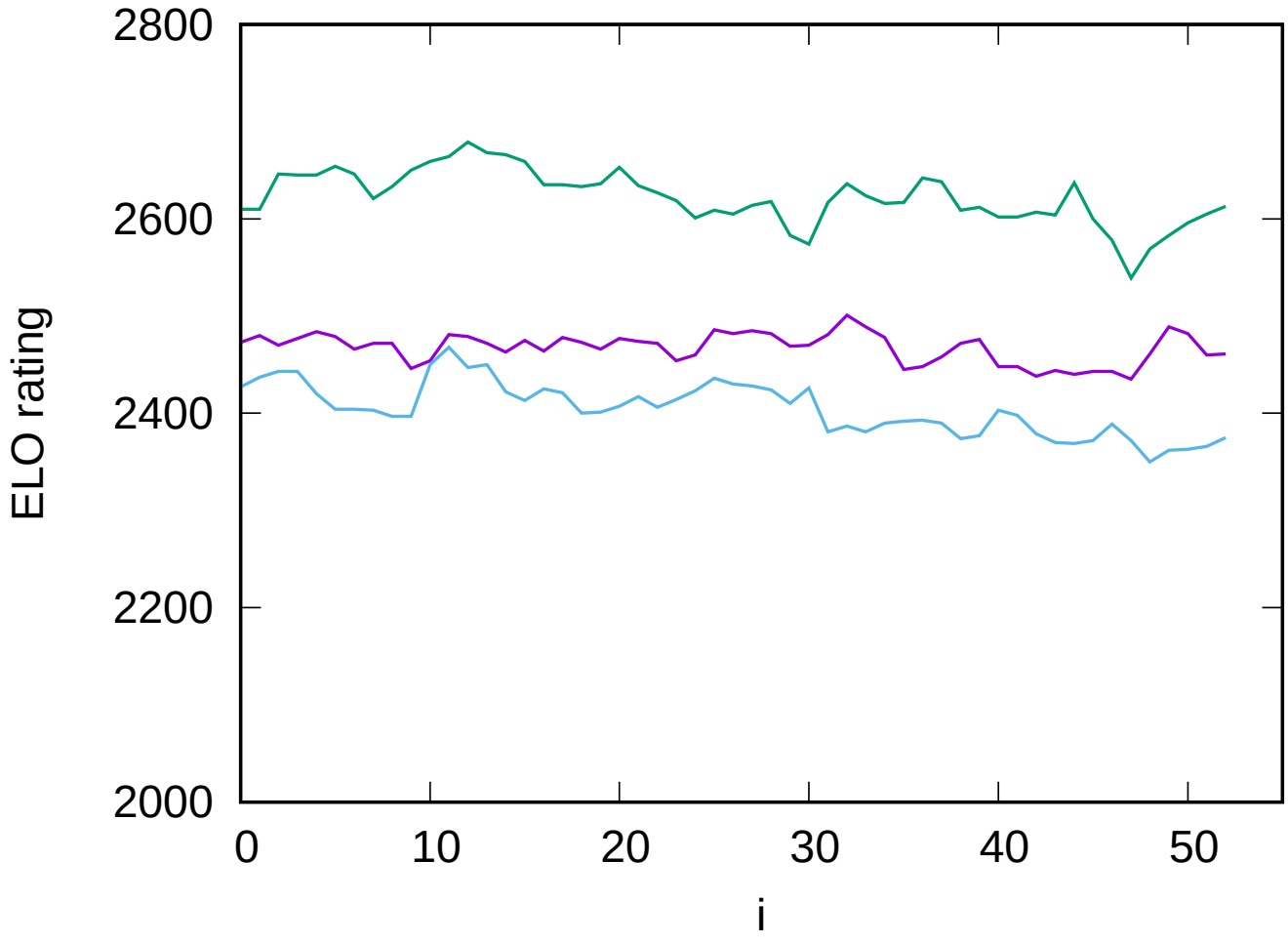

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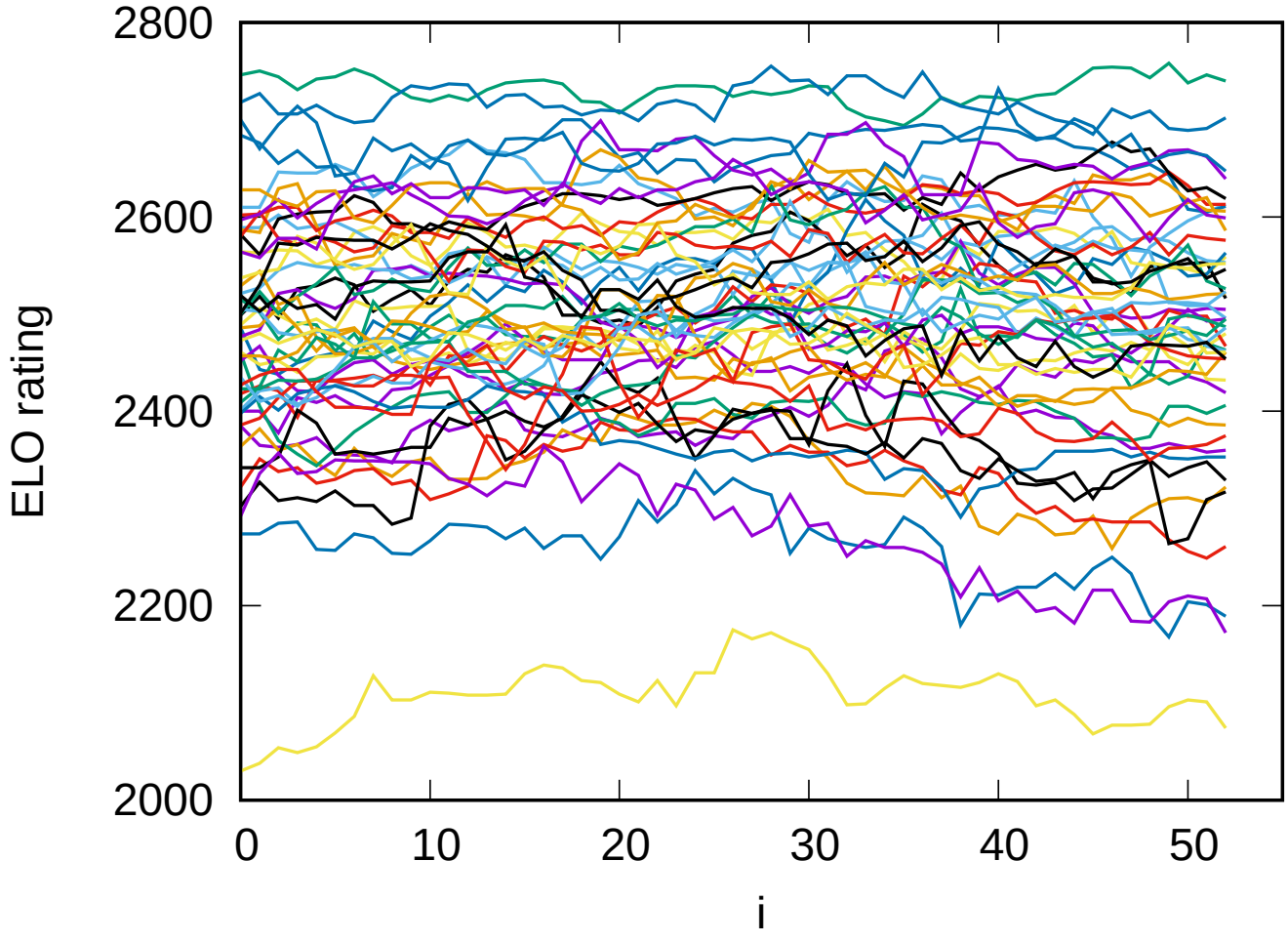
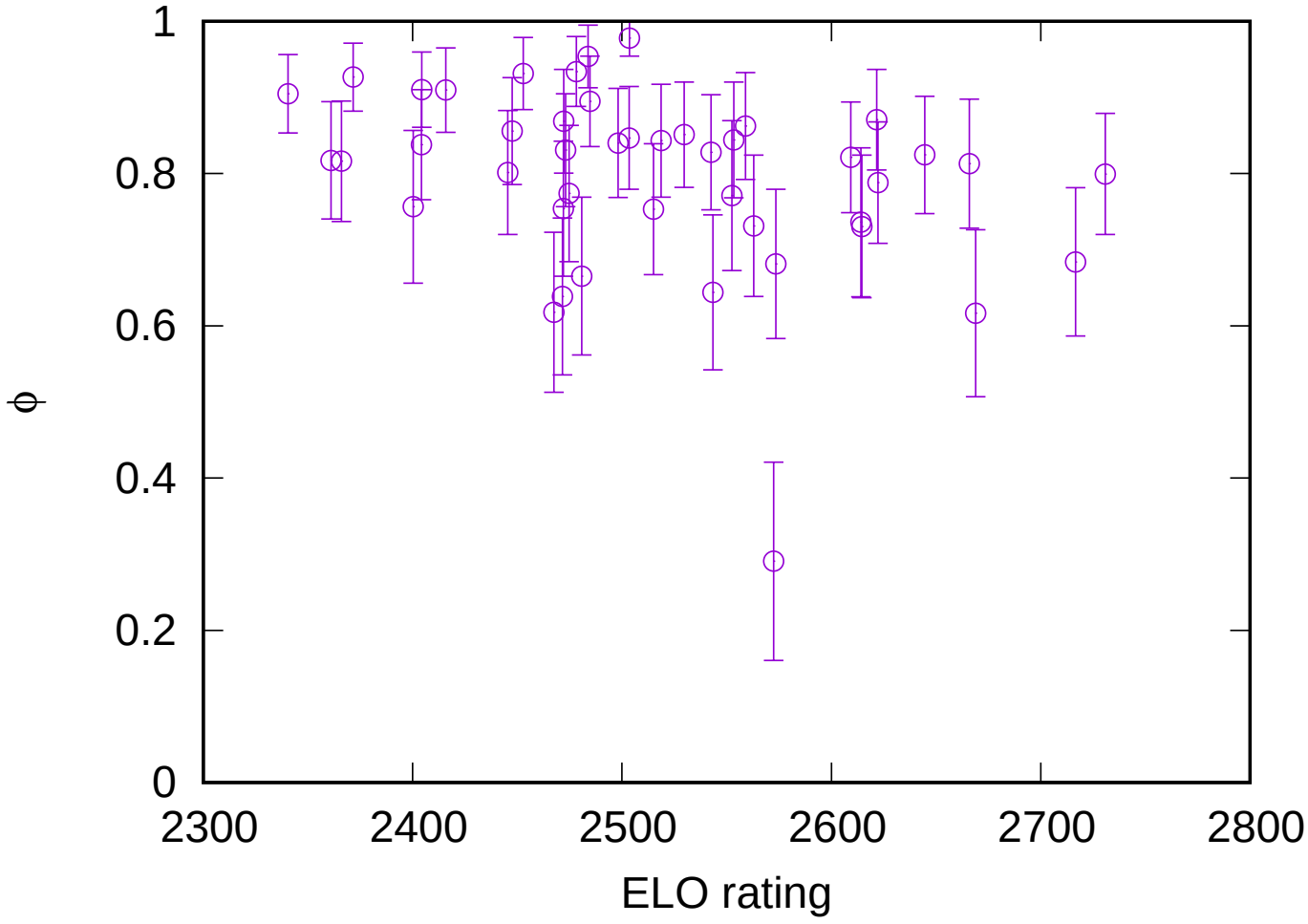


AR(1) analysis: autoregressive parameter ϕ

$Y_{n+\ell} = \phi Y_n + \sigma Z_{n+\ell}$



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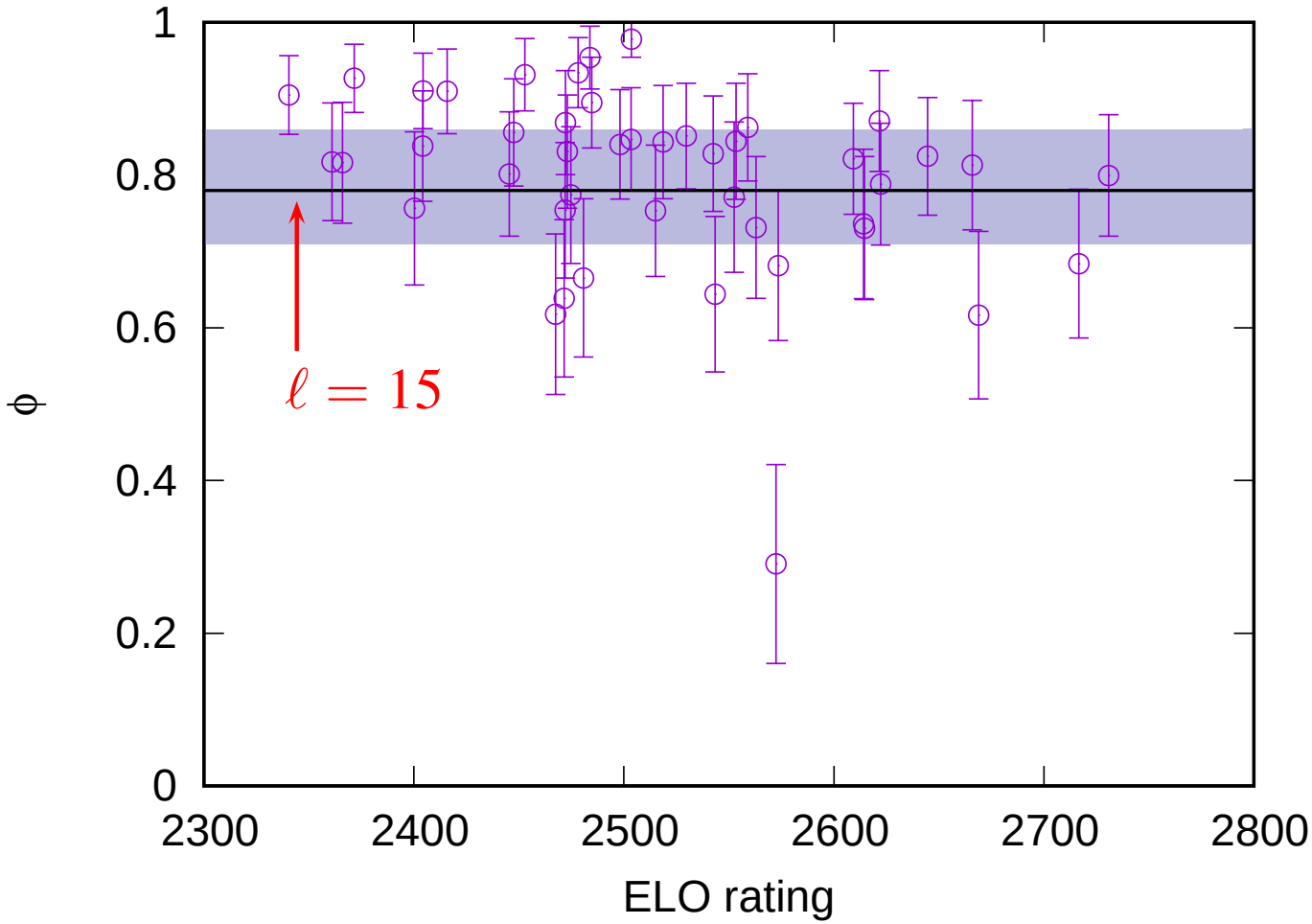

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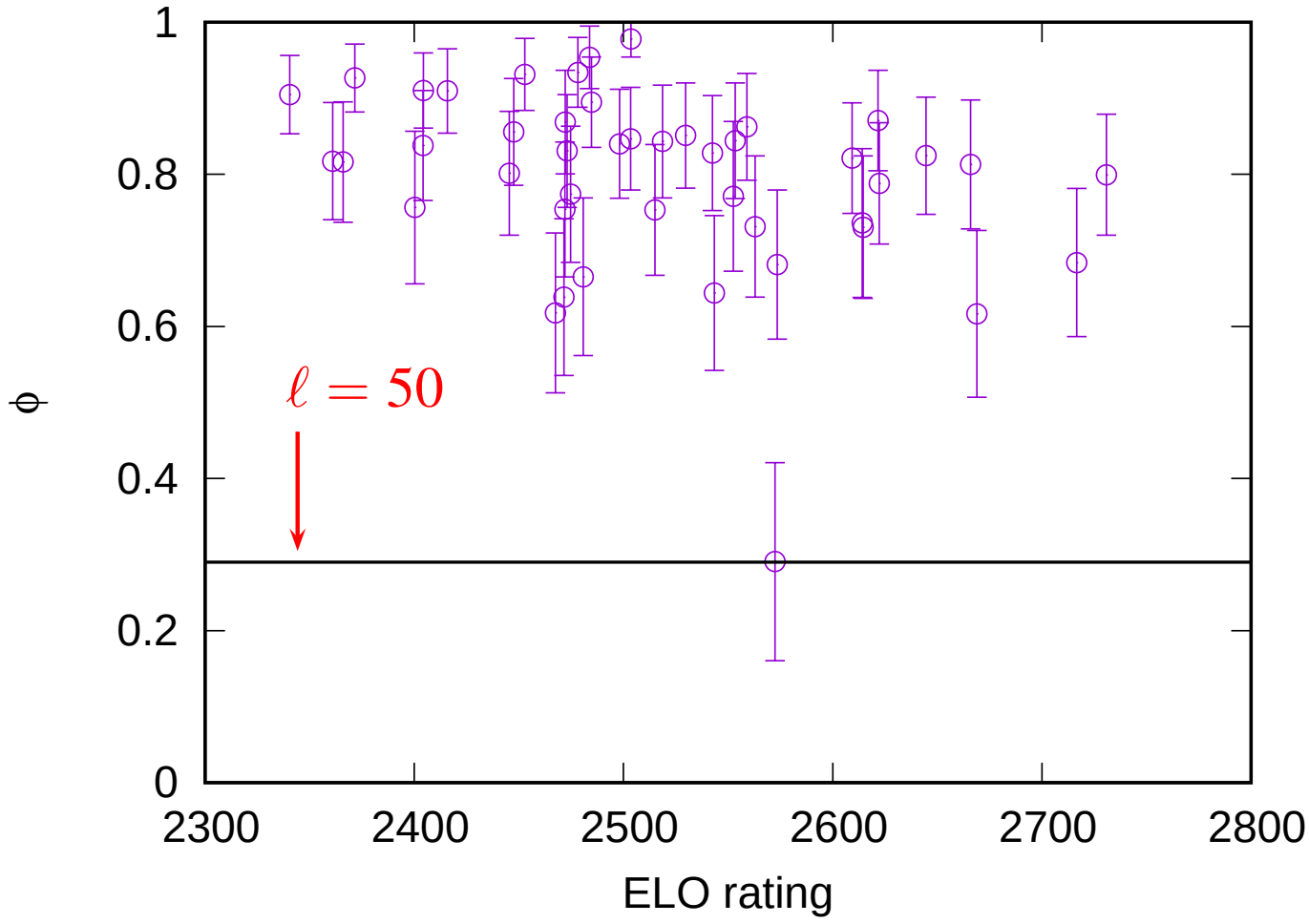

AR(1) analysis: noise amplitude σ

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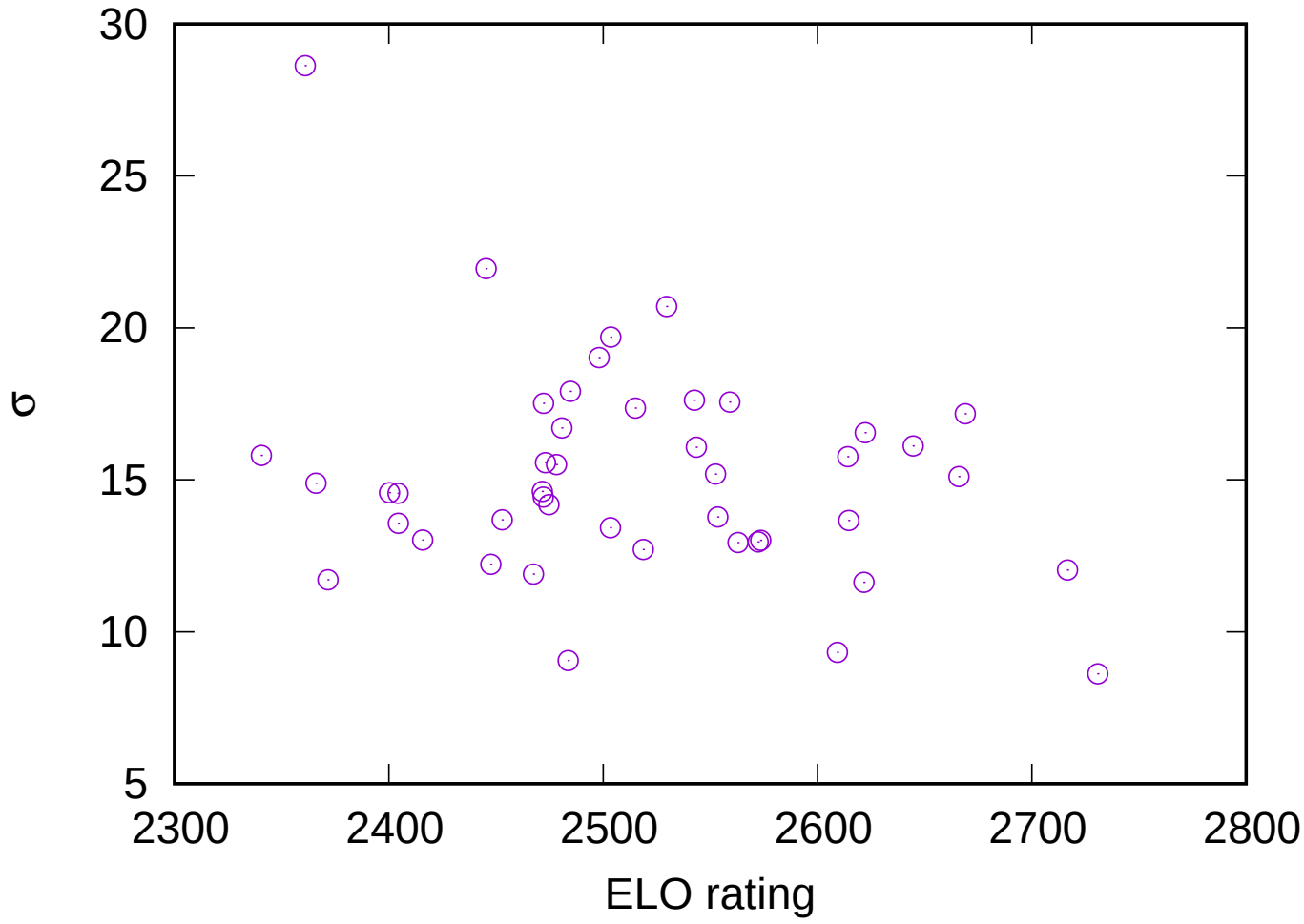

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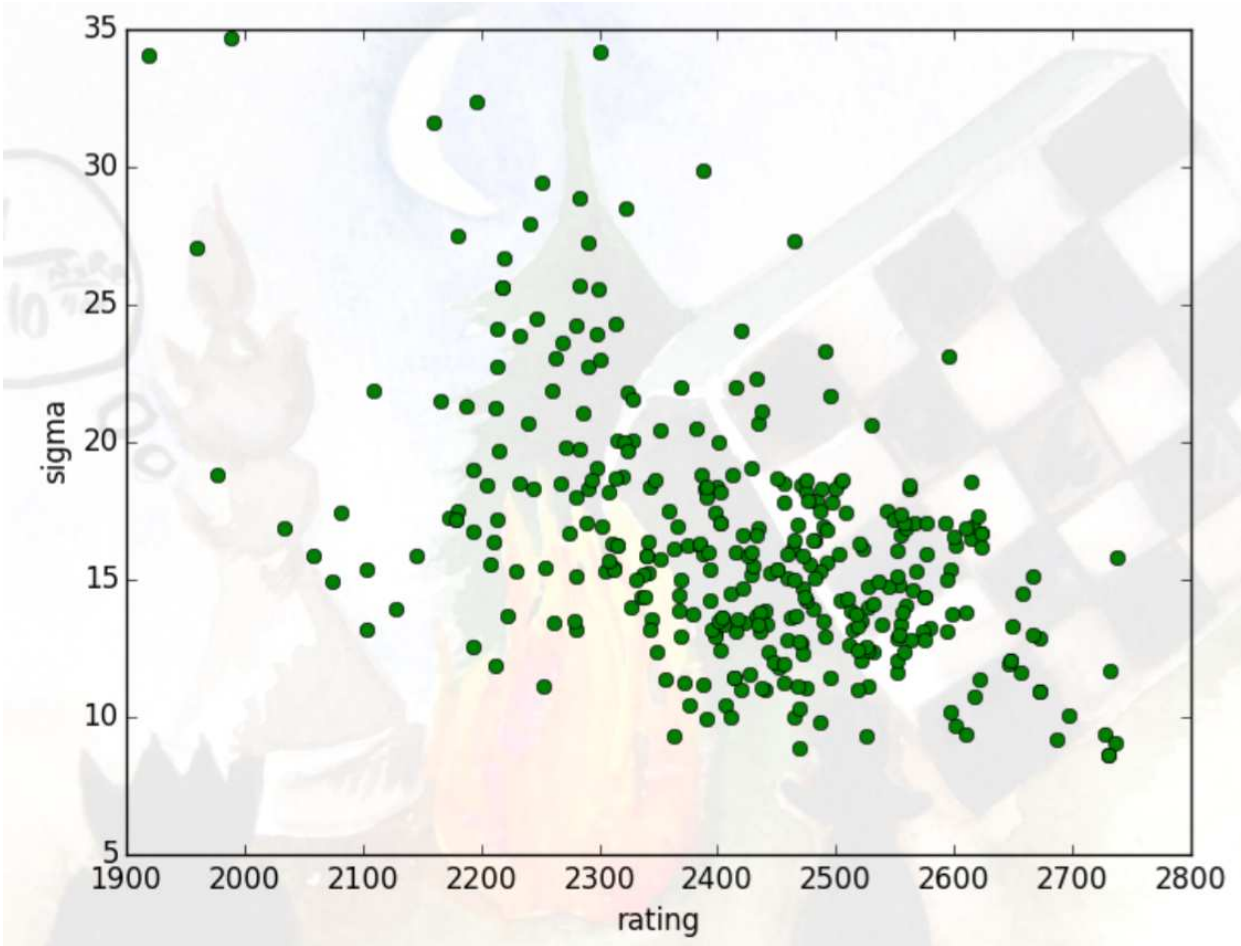

Summary

- ELO ratings are intended to be predictive.
- AR(1) approximates stationary rating trajectories.
- Computers may provide ‘objective’ ratings in future.

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