Stochastic bifurcation in random dynamical systems and its application to modeling atmospheric jet dynamics

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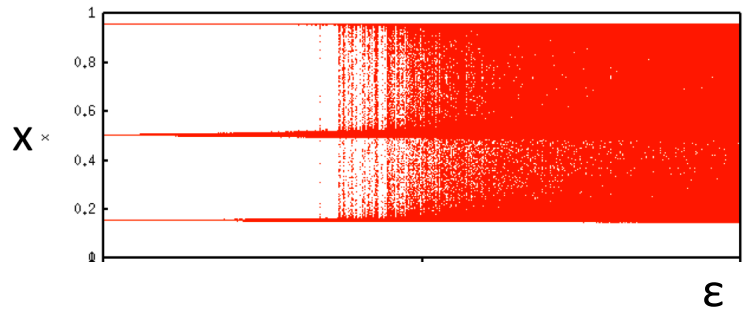
Random dynamical systems approaches to nonlinear stochastic phenomena

Random logistic map

$$x_{n+1} = a x_n (1 - x_n) + \xi_n$$

$$a = 3.83$$

$$\xi_n : \text{ bounded uniform noise in } [\epsilon, -\epsilon]$$



[G. Mayer-Kress and H. Haken, 1981] [YS, T-S Doan, M, Rasmussen, J. Lamb, in prep.]

Stochastic Lorenz equation

$$\begin{cases} dx = s(y - x)dt + \sigma x \, dW_t, \\ dy = (rx - y - xz)dt + \sigma y \, dW_t, \\ dz = (-bz + xy)dt + \sigma z \, dW_t. \end{cases}$$

r = 28, s = 10, b = 8/3 , σ = 0.3

Wt: Wiener process

[M. Chekroun, E. Simonnet, M. Ghil, 2011] [YS, M. Chekroun, M. Ghil, in prep.]

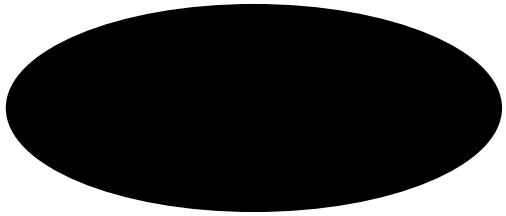
Outline

1. Random dynamical systems and stochastic chaos
2. Stochastic bifurcation in random logistic maps
3. Application: time series analysis for experimental data
4. Summary

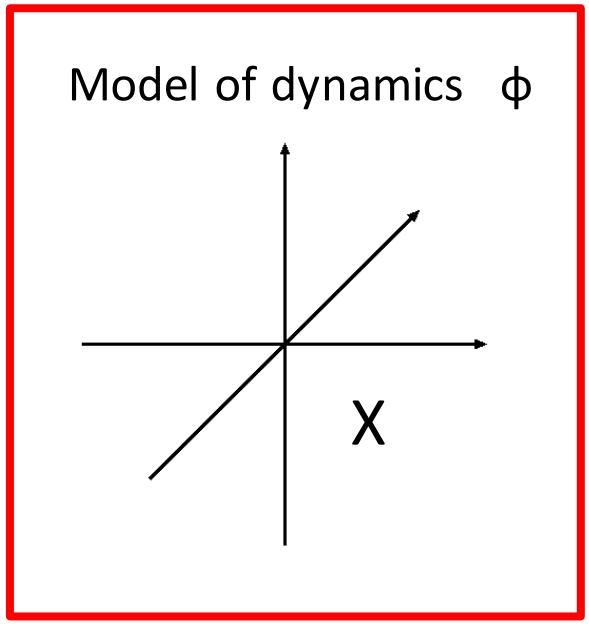
Random dynamical systems

A random dynamical system is the combination of two systems (θ, ϕ).

Model of noise θ



Random influence



Ω:{ω=(...,ω_0, ω_1, ω_2, ...)}

e.g. $\boxed{x_{n+1} = f(x_n) + \omega_n}$ ω_n: noise

State space

or $(\omega_n, x_n) = (\theta^n \omega_0,\ \phi(n, \theta^n \omega_0) x_0)$

$\Omega \times X$

Random attractor and its stability

Random attractor: A(ω)

An invariant random set of $x_{n+1} = f(x_n) + \xi_n = \phi(n, \omega)x_0$

satisfies $\lim_{n \to \infty} d(\phi(n, \theta^n \omega)B, A(\omega)) = 0$

for a bounded set B.

Random Lyapunov exponent: λ(ω)

$$\lambda(\omega, x) = \lim_{n \to \infty} \frac{1}{n} \log |\frac{\partial \phi(n, \omega)x}{\partial x}| \quad (x \in A(\omega))$$

Example: random point attractor

Langevin equation for Ornstein-Uhlenbeck process

$$dx = -\lambda x dt + \sigma dW_t \;\; (\lambda, \sigma > 0,\, W_t : Wiener\ process)$$

Random point attractor: $x(\omega)$

Invariant density: $\rho\big(x(\omega)\big) \sim \sqrt{\lambda/\pi\sigma^2}\, \exp\left(-\frac{\lambda x^2}{\sigma^2}\right)$

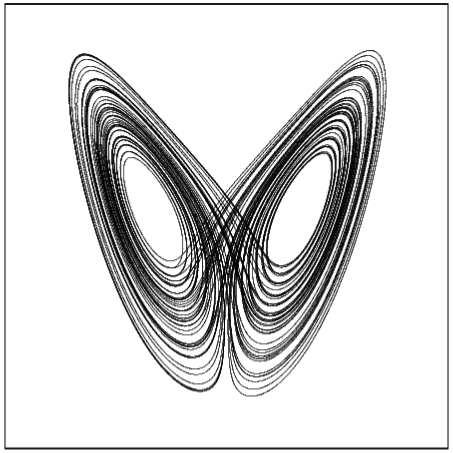
Lyapunov exponent: $-\lambda$

Example: random strange attractor

Lorenz system

$$dx/dt = s(y - x)$$
$$dy/dt = rx - y - xz$$
$$dz/dt = -bz + xy$$

r = 28, s = 10, b = 8/3



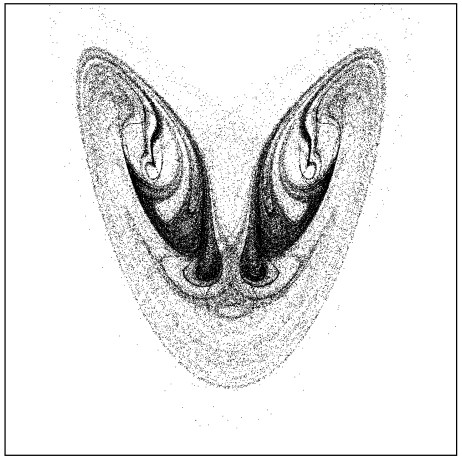
Strange attractor A:

1. Stable attractor
2. Stationary distribution
3. Positive top Lyapunov exponent

Stochastic Lorenz system

$$dx = s(y - x)dt + \sigma x dW_t$$
$$dy = (rx - y - xz)dt + \sigma y dW_t$$
$$dz = (-bz + xy)dt + \sigma z dW_t$$

r = 28, s = 10, b = 8/3 ,
σ = 0.3, Wt: Wiener process

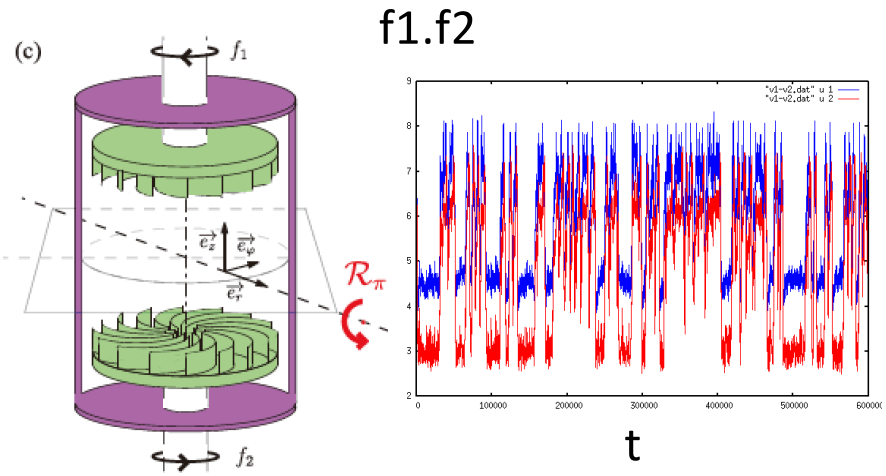


Random strange attractor A(ω):

1. Stable attractor
2. Stationary distribution
3. Positive top Lyapunov exponent

Stochastic chaos in a turbulent swirling flow

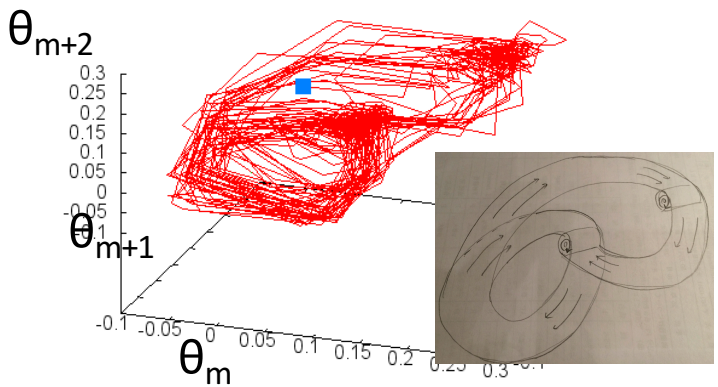
Collective motion in Karman flow



[B. Saint-Michel, et.al, 2013]

Time series embedding

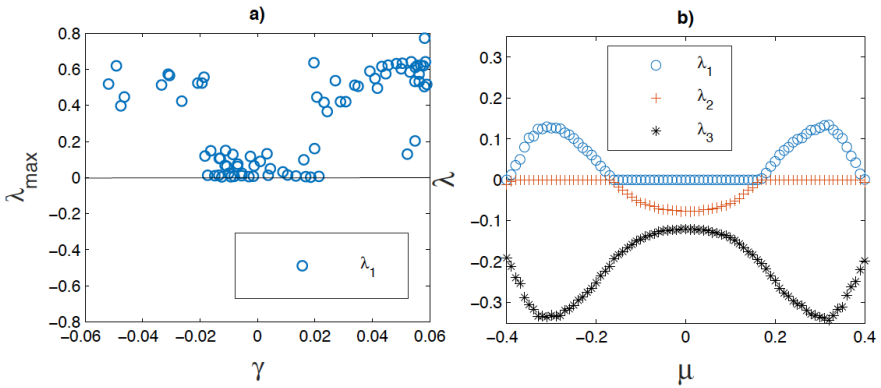
$$\theta = (f_1 - f_2)/(f_1 + f_2)$$



Model: Stochastic Duffing equation

$$\boxed{\begin{aligned} dx &= y dt \\ dy &= (-ay + x - x^3 + z\sin(\omega t))dt \\ dz &= -\phi(z-\mu)dt + \sigma dW_t \end{aligned}}$$

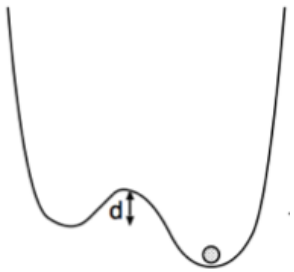
Lyapunov spectrum



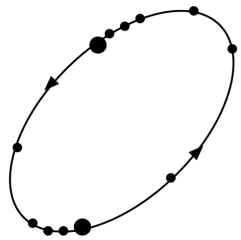
[D. Faranda, YS, B. Saint-Michel, C. Wiertel, V. Padilla, B. Dubrulle, F. Daviaud., PRL, 2017]

Noise-induced phenomena

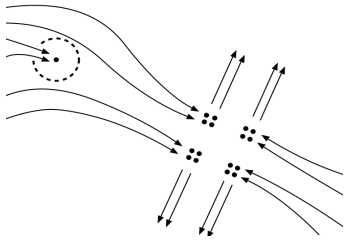
- Stochastic resonance [R. Benzi et. al., 1982]
  - Gradient dynamics
  - Potential barriers interact with noise
  - [A. Cherubini, J. Lamb, M. Rasmussen, and YS, 2017]



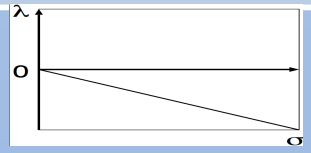
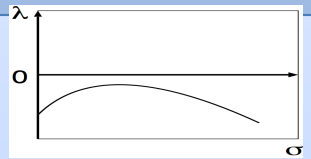
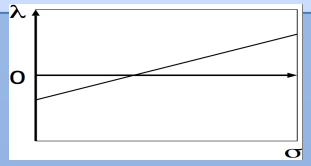
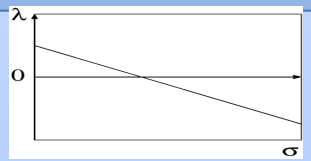
- Noise-induced synchronization [A. Pikovsky et. al., 1984]
  - Oscillatory dynamics
  - Stagnation points in phase interact with noise
  - [YS, and T.S. Doan, submitted]



- Noise-induced chaos [G. Mayer-Kress et. al., 1981]
  - Chaotic dynamics
  - Chaotic saddles, UPOs, interact with noise
  - [YS, M. Rasmussen, T.S. Doan, J. Lamb, to be submitted]



Random dynamical systems theory for noise-induced phenoma

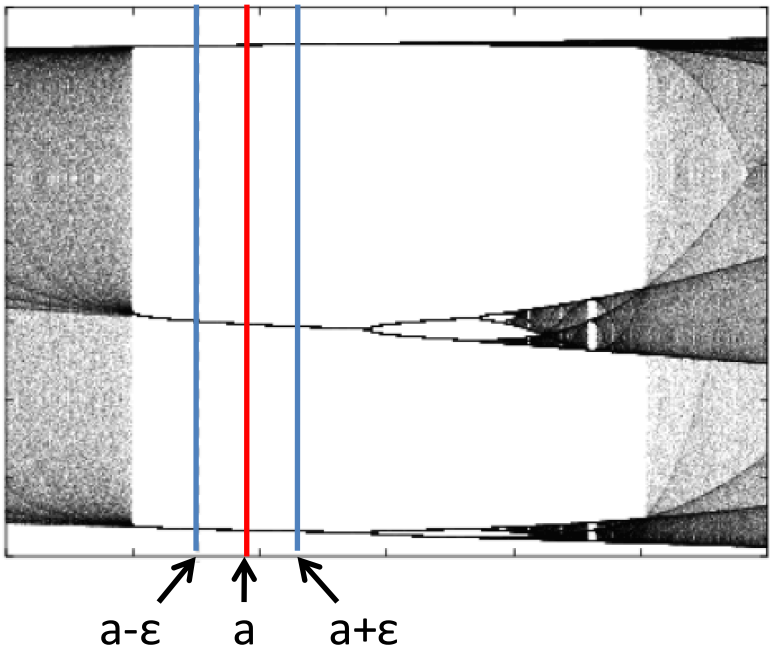
Noise-induced phenomena	**Stationary state**	**Topological bifurcation**	**Top Lyapunov exponent λ vs noise amplitude σ**
Noise-induced synchronization	random point attractor	Yes	
Stochastic resonance	random periodic attractor	No	
Noise-induced chaos	random strange attractor	Yes	
Noise-induced order	"window phenomena" weakly stationary	No	
Noise-induced intermittency	non-stationary Intermittency (infinite density)	Not at onset of topological bifurcation	λ=0

[A Cherubini, YS, M. Rasmussen, J. Lamb, 2017] [YS, R. Klages, submitted.]
[YS, T-S Doan, M, Rasmussen, J. Lamb, submitting] [YS, T.S. Doan, submitted]

Noise-induced chaos

Is period 3 logistic map in window region potentially chaotic in physical measurement?

Model: $x_{n+1} = a - x_n^2 + \epsilon\xi_n$ (a=1.755, ξ∈[-1,1]: noise)



Noise-induced chaos

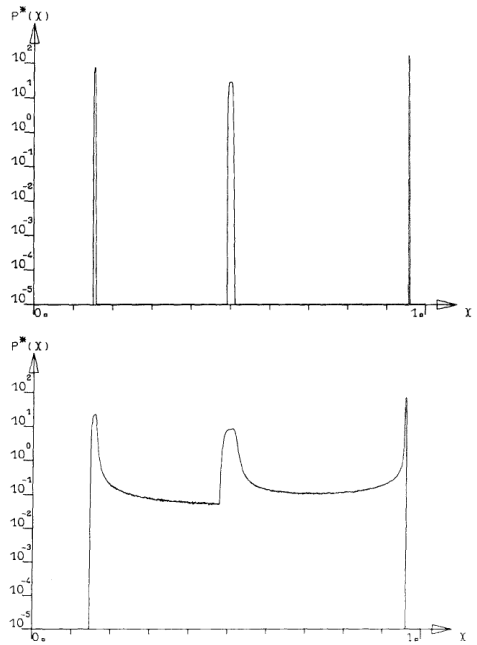
Small additive noise to period 3 window region makes non-attracting chaotic set observable.

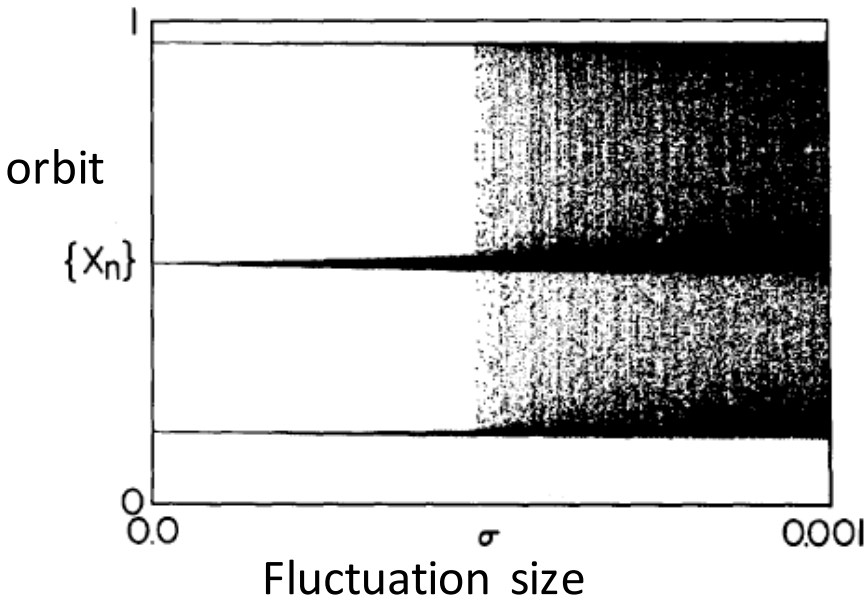
$$x_{n+1} = a - x_n^2 + \epsilon\xi_n$$ (a=1.755, ξ∈[-1,1]: noise)

Without noise

Invariant densities

With noise

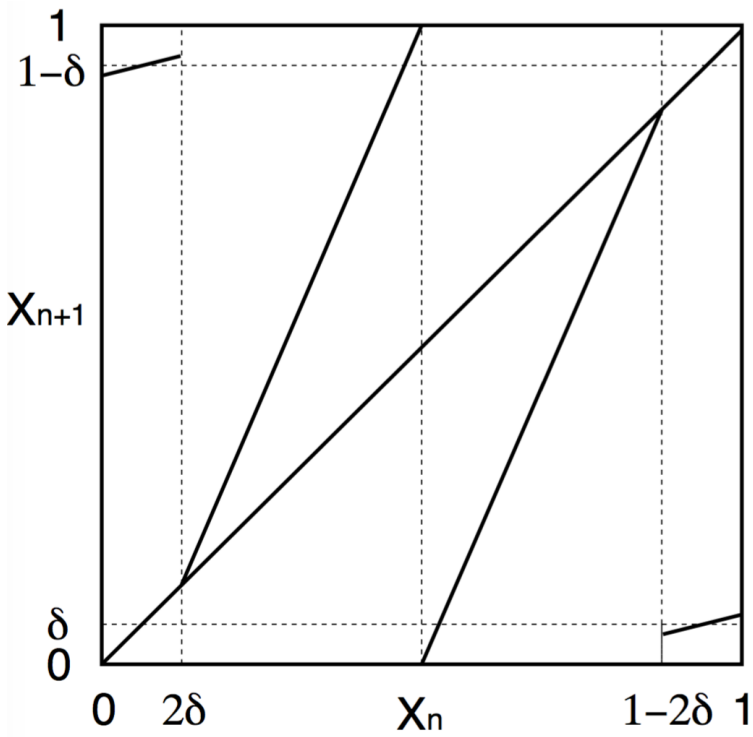




[Crutchfield et.al., 1982]

Noise-induced chaos

Chaotic transient

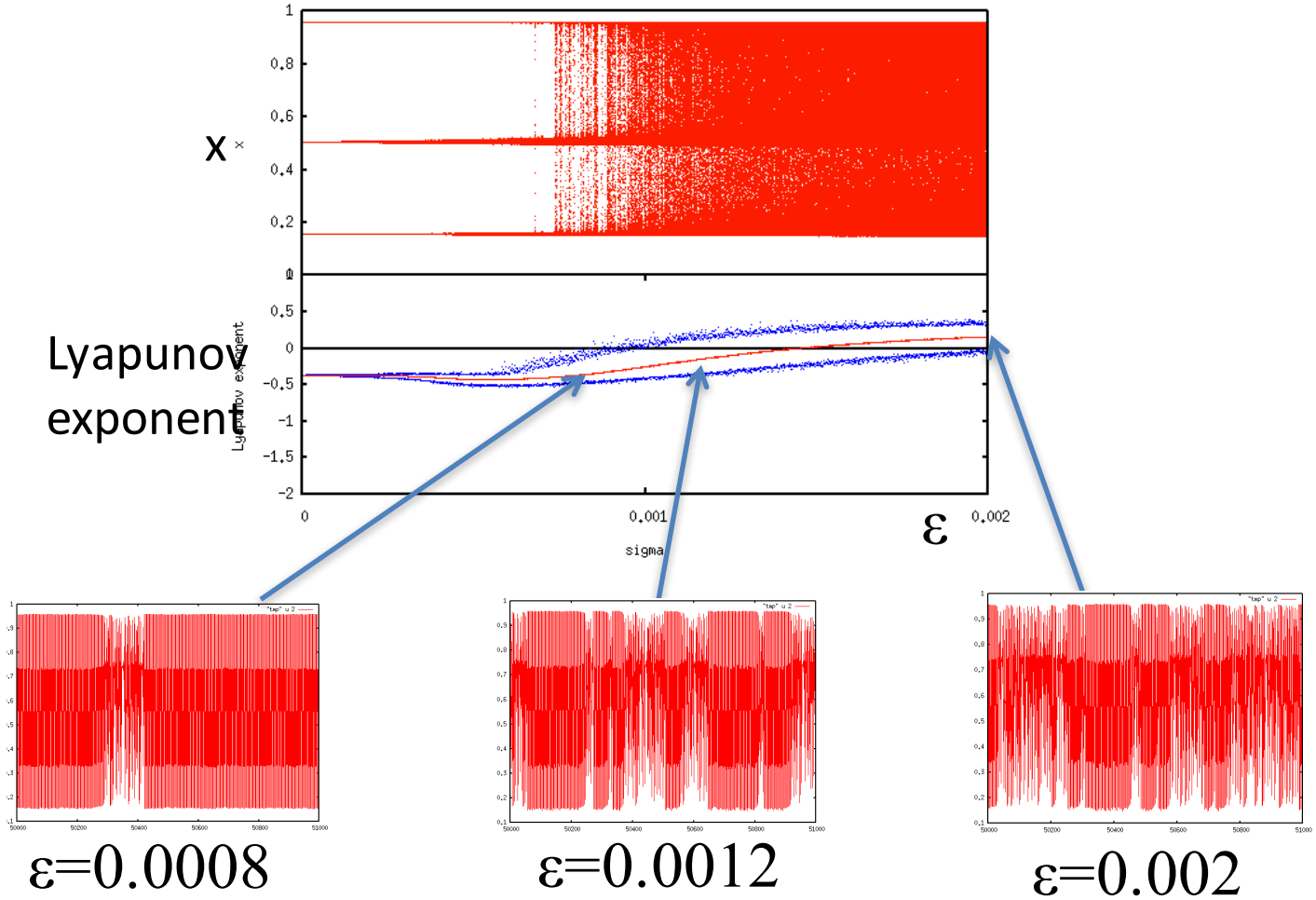


Stable limit cycle

Noise-induced chaos

$$x_{n+1} = a x_n (1 - x_n) + \xi_n$$

$$a = 3.83 \quad \xi_n : \text{ bounded uniform noise in } [\epsilon, -\epsilon]$$



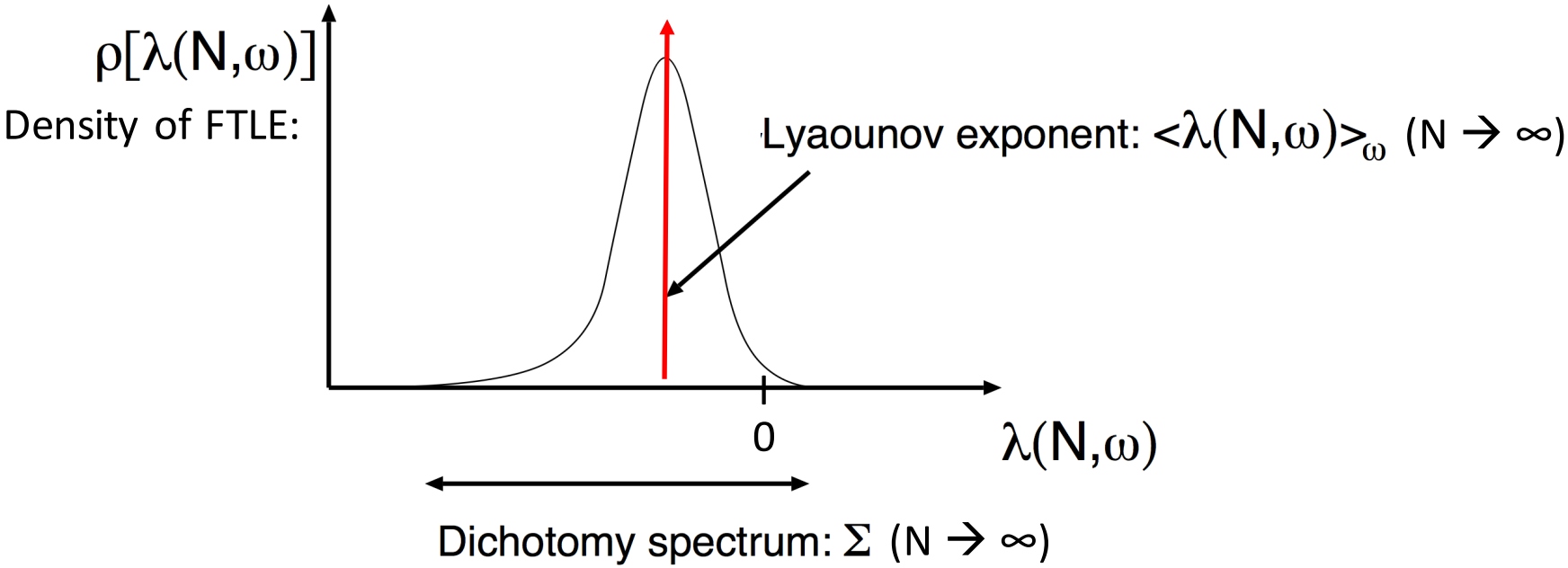
Dichotomy spectrum

K>0, α>0 $$Ke^{-(\gamma+\alpha)n}|x| \leq |\frac{\partial\phi}{\partial x}(n,\omega,x)| \leq Ke^{(\gamma-\alpha)n}|x|,$$

for all n, for almost all ω

Dichotomy spectrum is given by $\Sigma = \cup\{\gamma\}$

Finite time Lyapunov exponent $\lambda(N,\omega)$



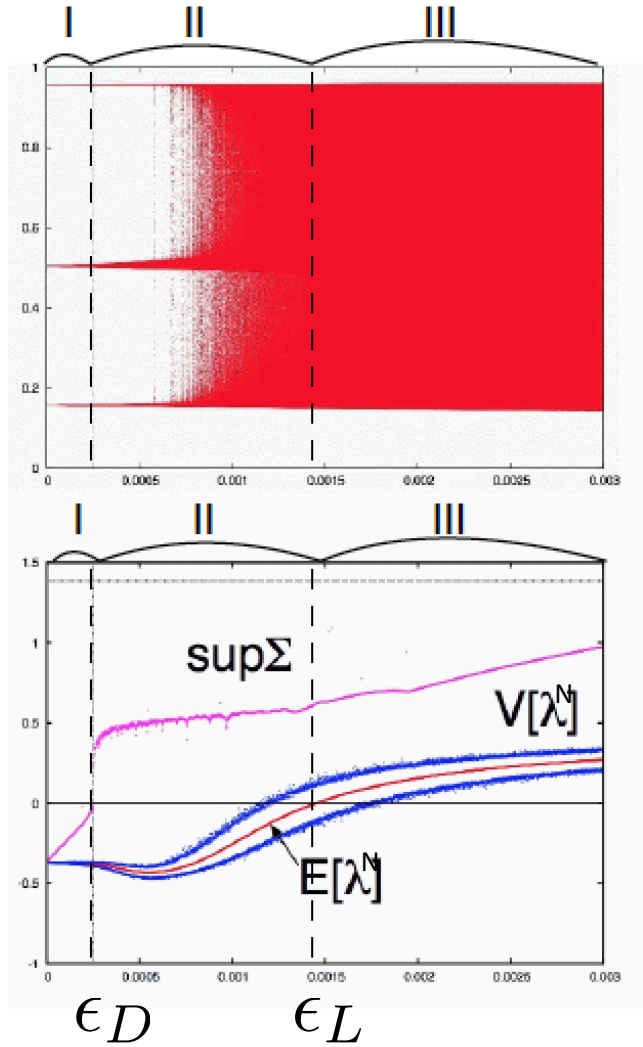
[M. Callaway, T. S. Doan, J. S. Lamb, and M. Rasmussen, 2016]

Lyapunov exponent and dichotomy spectrum

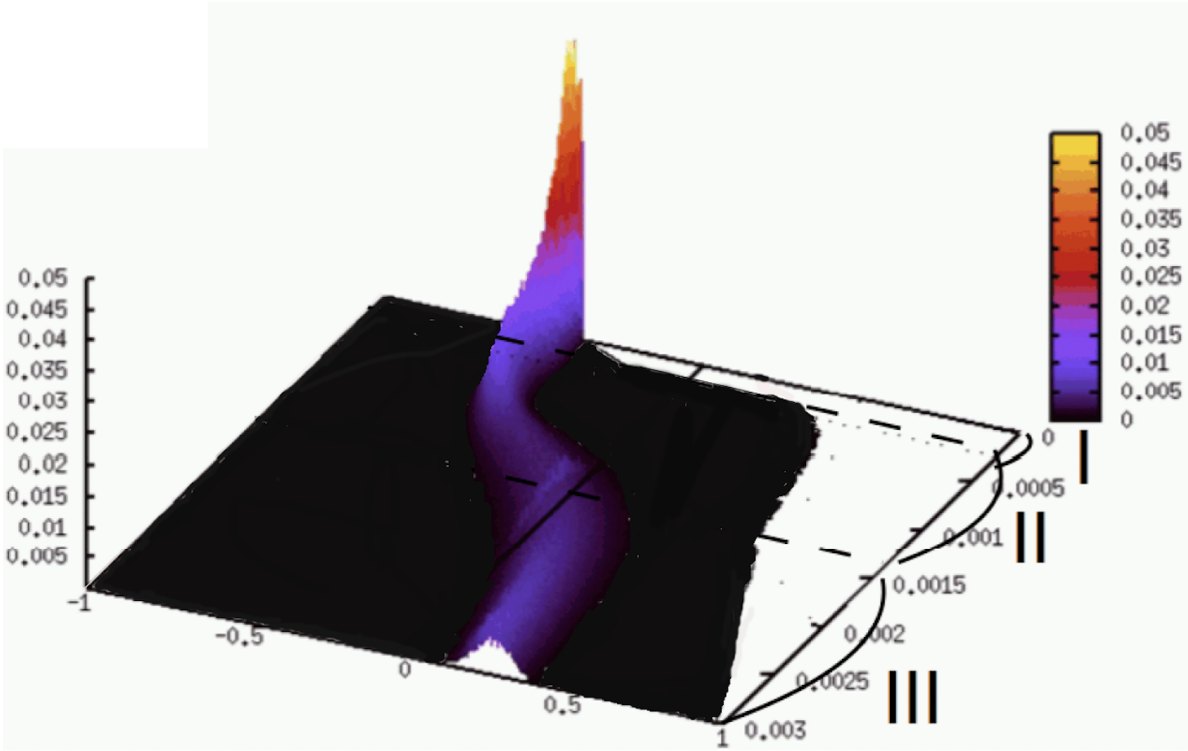
$$x_{n+1} = f_a(x_n) + \xi_n \qquad \xi_n \in [-\epsilon, \epsilon]$$

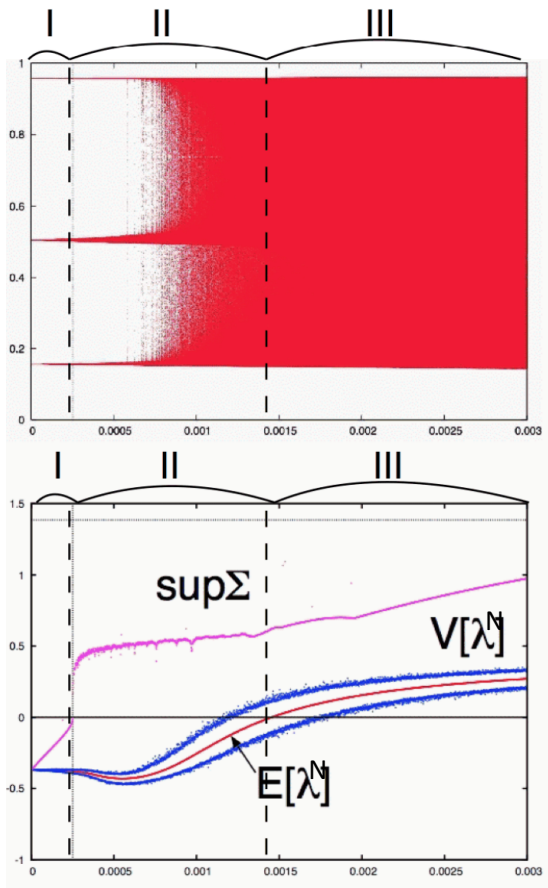
ϵ_D : Topological bifurcation point

ϵ_L : Transition point of stability of random attractor



Distribution of finite time Lyapunov exponents





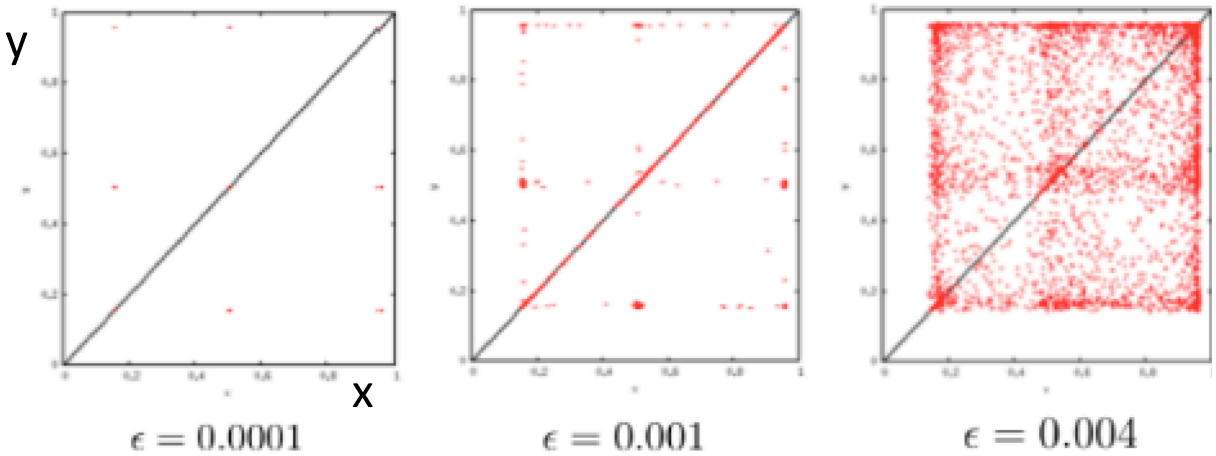
ϵ_D ϵ_L

A route to stochastic chaos

$$x_{n+1} = f_a(x_n) + \xi_n$$
$$y_{n+1} = f_a(y_n) + \xi_n$$

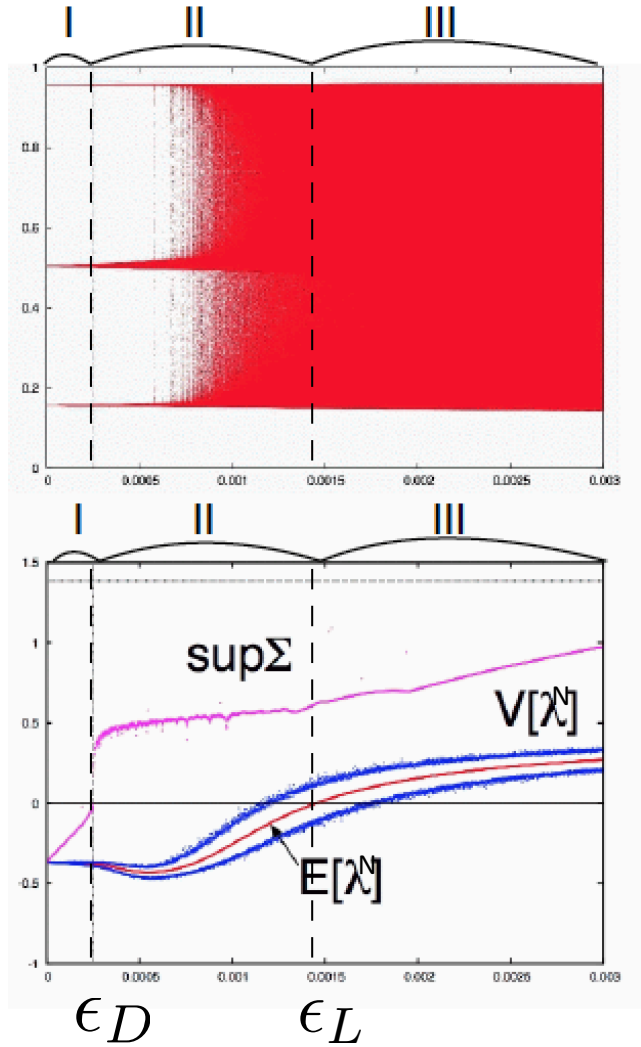
$$\xi_n \in [-\epsilon, \epsilon]$$

Phase I	Phase II	Phase III
(Noised limit cycle)	(Partially chaotic)	(Stochastic chaos)



λ<0, supΣ<0 → λ<0, supΣ>0 → λ>0, supΣ>0

Random periodic attractor | Random point attractor | Random strange attractor



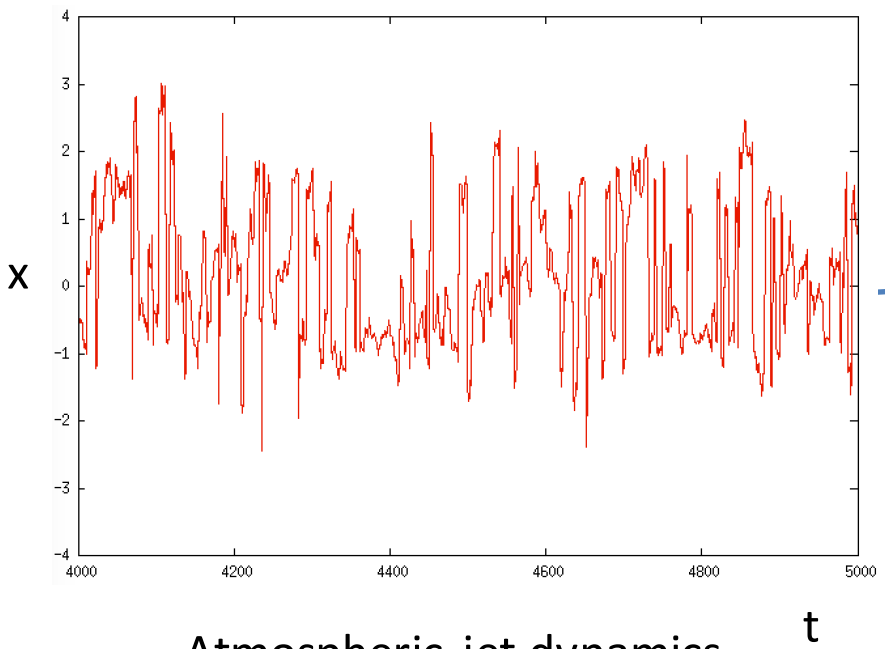
[YS, M. Rasmussen, T.S. Doan, J. Lamb, to be submitted]

Summary

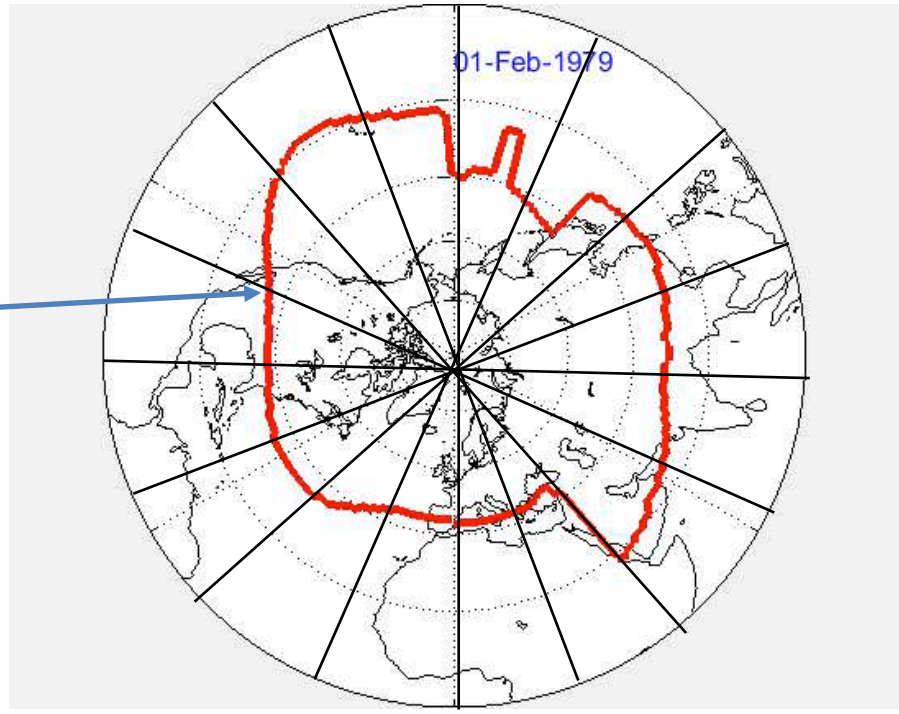
1. Zero-crossing point of Lyapunov exponent determines transition point of stability of random attractors. (most likely asymptotic behaviour)
2. Zero-crossing point of supremum of dichotomy spectrum determines topological bifurcation points. (all possible asymptotic behaviour)

Application: blocking phenomena in atmospheric jet dynamics

Generation and annihilation of kink and anti-kinks
(Blocking phenomena)



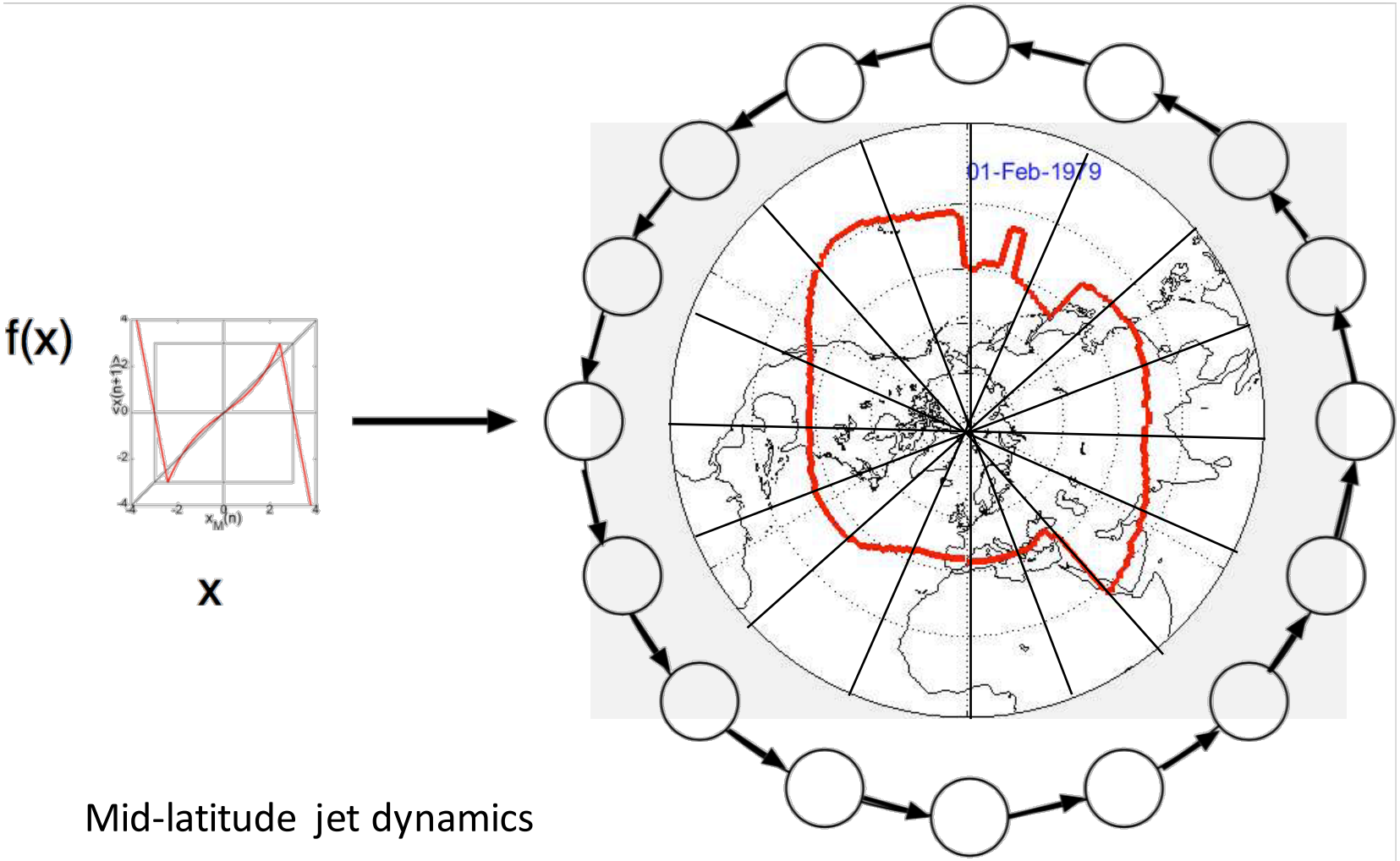
Atmospheric jet dynamics at Rocky Mountains for 3 years observed by day



mid-latitude jet dynamics

[with D. Faranda, G. Messori, N. Moloney, Y. Pascal, to be submitted.]

The model for global dynamics



Mid-latitude jet dynamics

$$x_{n+1}^{(i)} = (1-\epsilon) f(x_n^{(i)}) + \epsilon f(x_n^{(i-1)}) + \xi_n^{(i)}, \quad \xi_n^{(i)} = \nu^{(i)} + r^{(i)} + \eta^{(i)}$$

The model for global dynamics

$$x_{n+1}^{(i)} = (1-\epsilon) f(x_n^{(i)}) + \epsilon f(x_n^{(i-1)}) + \xi_n^{(i)},$$

$$\xi_n^{(i)} = \nu^{(i)} + r^{(i)} + \eta^{(i)}$$