Madogram and asymptotic independence among maxima

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Abstract:

- A strong statistical research effort has been devoted in multivariate extreme value theory in order to assess the strength of dependence among extremes. This topic is particularly difficult in the case where block maxima are near independence. In this paper, we adapt and study a simple inference tool inspired from geostatistics, the madogram, to the context of asymptotic independence between pairwise block maxima. In particular, we introduce an extremal coefficient and study the theoretical properties of its estimator. Its behaviour is also illustrated on a small simulation study and a real data set.

Key-Words:

- *Extremal coefficient; method-of-moment; maximum likelihood.*

AMS Subject Classification:

- 62G05, 62G20, 62G32.

1. INTRODUCTION

One recurrent question in multivariate extreme value theory (MEVT) is how to infer the strength of dependence among maxima. To illustrate this inquiry by an example, monthly maxima of hourly precipitation measured at two french stations from February 1987 to December 2002 are displayed in Figure 1. The two stations belong to the same hydrological basin of Orgeval (`https://gisoracle.cemagref.fr/`) that is located in France, west of Paris.

For each season, a scatterplot between the two stations shows the original 45 (15 years $\times$ 3 months per season) monthly maxima recorded in millimeters. The dependence structure seems to vary according to seasons and it is not clear if the largest summer values are close to independence.

This concept of asymptotic independence has been studied by many authors. In this paper, we follow the approach introduced by Ledford and Tawn (1996) and extended by Ramos and Ledford (2009). Before explaining the details of our method, we need to recall a few basic concepts about MEVT and to introduce some notations. Suppose that we have at our disposal n independent and identically distributed pairs (X_i, Y_i) with unit-Fréchet margins ($\mathbb{P}(X_i \leq x) = \exp(-1/x)$ for $x > 0$) and that the component-wise maxima vector $(M_{X,n}, M_{Y,n}) = (\max(X_1, \dots, X_n), \max(Y_1, \dots, Y_n))$ converges in the following way:

$$\text{(1.1)} \qquad \lim_{n\to\infty} \mathbb{P}\left(\frac{M_{X,n}}{n} \leq x, \frac{M_{Y,n}}{n} \leq y\right) = G(x,y), \text{ for } x, y > 0.$$

The limiting distribution function G is called the bivariate extreme value distribution and it can be written as $G(x,y) = \exp\{-V(x,y)\}$, with

$$V(x,y) = \int_0^1 \max\left(\frac{\omega}{x}, \frac{1-\omega}{y}\right) dH(\omega),$$

where $H(.)$ is a finite non-negative measure on $[0,1]$ such that $\int_0^1 \omega dH(w) = \int_0^1 (1-\omega) dH(w) = 1$. This latter condition on H ensures that the margins $G(x,\infty)$ and $G(\infty,y)$ are unit-Fréchet distributed. The function V is called the pairwise extremal dependence function. It is homogeneous of order -1, i.e. $V(tx,ty) = t^{-1}V(x,y)$ for any positive t and G is max-stable, i.e. $G^t(tx,ty) = G(x,y)$. By definition of H, the function V has no explicit form and various non-parametric estimators of V have been studied (e.g. Capéraà *et al.*, 1997). As an example, an approach based on a classical geostatistical tool, the madogram (Matheron, 1987), was proposed by Naveau *et al.* (2009). Its simplest version (Cooley *et al.*, 2006) focused on the estimation of the extremal coefficient $\theta = V(1,1)$. This coefficient provides a quick summary of the dependence between maxima. It belongs to the interval $[1,2]$. If θ equals two, the pairwise maxima are independent, and if θ equals one, it is the complete dependence case. Cooley *et al.* (2006) defined the so-called F-madogram

$$\nu = \frac{1}{2}\mathbb{E}\left|F\left(M_{X,n}\right) - F\left(M_{Y,n}\right)\right|, \tag{1.2}$$

where F denotes the distribution function of $M_{X,n}$ and $M_{Y,n}$, in order to express the extremal coefficient as

$$\theta = \frac{1+2\nu}{1-2\nu}. \tag{1.3}$$

Going back to the maxima displayed in Figure 1, one may wonder if convergence (1.1) provides an appropriate probabilistic framework to study the near independence seen in our summer rainfall data. Convergence (1.1) implies that $\lim n\mathbb{P}\left(\frac{X_i}{n} > x \text{ or } \frac{Y_i}{n} > y\right) = -\log G(x,y)$. Hence

$$\lim_{n\longrightarrow\infty} n\mathbb{P}\left(\frac{X_i}{n} > x \text{ and } \frac{Y_i}{n} > y\right) = \log G(x,y) - \log G(x,\infty) - \log G(\infty,y).$$

If we are in the asymptotically independent case, i.e. $G(x,y)$ can be written as the product $G(x,y) = G(x,\infty)G(\infty,y)$, the right-hand side of the last convergence is nothing else than zero. This result is uninformative about the degree of dependence among our rainfall maxima. A conceptual extension is needed to improve our understanding of the probability of having joint extremes. To fill in this gap, Ledford and Tawn in a series of papers (e.g. Ledford and Tawn, 1996, 1997, 1998) introduced a new tail model of the distribution which has been simplified by Ramos and Ledford (2009) as follows

$$\mathbb{P}\left(X > x, Y > y\right) = (xy)^{-\frac{1}{2\eta}}\mathcal{L}(x,y), \text{ for some } \eta \in (0,1], \tag{1.4}$$

with $\mathcal{L}$ a bivariate slowly varying function at infinity. The coefficient of tail dependence, η, is a measure of asymptotic independence. It is equal to one in the asymptotic dependence case and less than one in the asymptotic independence one. Condition (1.4) is tailored to analyze large excesses in the asymptotic independent case but it needs a reformulation in order to be used with pairs of maxima, as the ones pictured in Figure 1. This reformulation has been recently proposed by Ramos and Ledford (2011) who studied an extension of (1.1) by proving under the tail model (1.4) that, for $x, y > 0$,

$$\lim_{\varepsilon\to 0}\lim_{n\to\infty} \mathbb{P}\left[\frac{M_{X,n,\varepsilon b_n}}{b_n} \le x, \frac{M_{Y,n,\varepsilon b_n}}{b_n} \le y\right] = G_\eta(x,y) = \exp\Big[-V_\eta(x,y)\Big], \tag{1.5}$$

where the normalising constants b_n are defined implicitly as $n\mathbb{P}(X > b_n, Y > b_n) = 1$, $M_{\bullet,n,\varepsilon}$ corresponds to the component-wise maxima such that (X_i, Y_i) occur within the set $R_\varepsilon = \{(x,y) : x > \varepsilon, y > \varepsilon\}$ and

$$V_\eta(x,y) = \eta\int_0^1 \left[\max\left(\frac{\omega}{x}, \frac{1-\omega}{y}\right)\right]^{\frac{1}{\eta}} dH_\eta(\omega), \tag{1.6}$$

with H_η a finite and non-negative measure satisfying the constraint

$$\eta^{-1} = \int_0^{\frac{1}{2}} \omega^{\frac{1}{\eta}} dH_\eta(\omega) + \int_{\frac{1}{2}}^1 (1-\omega)^{\frac{1}{\eta}} dH_\eta(\omega).$$

The new dependence function V_η is homogeneous of order $-\frac{1}{\eta}$:

$$V_\eta(tx, ty) = t^{-\frac{1}{\eta}} V_\eta(x, y),$$

and the distribution $G_\eta(x, y)$ obeys an extended max-stable property:

$$G_\eta^{t^{1/\eta}}(tx, ty) = G_\eta(x, y).$$

In (1.1), a normalisation of n^{-1} is required in order to stabilize the componentwise maxima whereas in (1.5) b_n is of order $O(n^\eta)$.

The main goal of this paper is to adapt the concept of madogram to this framework of asymptotic independence. The asymptotic properties of our estimators are also derived. A small simulation study allows us to compare our inference scheme with the maximum likelihood estimation approach. All these estimators are applied to our rainfall data set.

2. THE F-MADOGRAM IN THE ASYMPTOTIC INDEPENDENCE CASE

Denote (M_X^*, M_Y^*) the bivariate vector that follows the distribution $G_\eta(x, y)$, i.e.

$$\mathbb{P}\left(M_X^* \le x, M_Y^* \le y\right) = \exp\{-V_\eta(x, y)\}, \tag{2.1}$$

with $V_\eta(x, y)$ of the form (1.6).

Concerning the marginals, we denote

$$F_X^*(x) := \mathbb{P}\left(M_X^* \le x\right) = \exp\Big[-\sigma_X x^{-\frac{1}{\eta}}\Big] \text{ and } F_Y^*(y) := \exp\Big[-\sigma_Y y^{-\frac{1}{\eta}}\Big], \tag{2.2}$$

with $\sigma_X = V_\eta(1, \infty)$ and $\sigma_Y = V_\eta(\infty, 1)$. As the scaling coefficients σ_X and σ_Y are not necessarily equal, the Fréchet margins of M_X^* and M_Y^* differs by a multiplicative factor. In the classical MEVT setup defined by (1.1), the extremal coefficient $\theta = V(1, 1)$ was simple to explain. It always varied between one (dependence) and two (independence). Having different marginals in (2.2) makes it difficult to find simple and interpretable summaries like the extremal coefficient. One possible way around this interpretability issue is to go back to the madogram distance because it is trivial to interpret it as a metric and it is marginal free. The F-madogram for the pair (M_X^*, M_Y^*) can be defined as

$$\nu_\eta := \frac{1}{2}\mathbb{E}\left|F_X^*\left(M_X^*\right) - F_Y^*\left(M_Y^*\right)\right|, \tag{2.3}$$

and we can derive from (1.6) and (2.2) the relationship (see Appendix)

$$\theta_\eta = \frac{1+2\nu_\eta}{1-2\nu_\eta}, \tag{2.4}$$

where $\theta_\eta := V_\eta(\sigma_X^\eta, \sigma_Y^\eta)$ could be viewed as an analog of the classical extremal coefficient, comparing equations (1.3) and (2.4). If ν_η equals zero, then θ_η equals one. As the distance ν_η increases, the coefficient θ_η also increases. If M_X^* and M_Y^* are independent, then $F_X^*\left(M_X^*\right)$ and $F_Y^*\left(M_Y^*\right)$ are independent and uniformly distributed random variables. It follows that $\nu_\eta = 1/6$. From (2.4), we deduce that $\theta_\eta = 2$.

The only difference between equations (1.3) and (2.4) resides in the fact that the pairwise maxima vector belongs now to the largest family G_η instead of the classical G. It is also essential to emphasize that the F-madogram should not be interpreted alone. The coefficient η is paramount to explore the asymptotic independence domain.

3. INFERENCE

3.1. A method-of-moment approach

Our main result is the following theorem that deals with the convergence of the empirical estimator deduced from (2.3).

Theorem 1 *Let $\left(M^*_{X_i,n}, M^*_{Y_i,n}\right)$ be a sample of N bivariate maxima vectors of block size n that converges in distribution, as $n \to \infty$, to a bivariate extreme value distribution with an $\eta-$dependence function defined as in (1.6). Let*

$$\widehat{\nu}_\eta = \frac{1}{2N}\sum_{i=1}^{N}\left|\widehat{F}_X^*(M^*_{X_i,n}) - \widehat{F}_Y^*(M^*_{Y_i,n})\right|, \tag{3.1}$$

where $\widehat{F}_X^*$, resp. $\widehat{F}_Y^*$, denotes the empirical distribution function of the sample $M^*_{X_i,n}$, resp. $M^*_{Y_i,n}$. Then, as $n \to \infty$ and $N \to \infty$

$$\sqrt{N}\left(\widehat{\nu}_\eta - \nu_\eta\right) \xrightarrow{d} \int_{[0,1]^2} N_C(u,v)dJ(u,v),$$

where $J(x,y) = \frac{1}{2}|x-y|$ and N_C is a Gaussian process defined by

$$N_C(u,v) = B_C(u,v) - B_C(u,1)\frac{\partial C}{\partial u}(u,v) - B_C(1,v)\frac{\partial C}{\partial v}(u,v), \tag{3.2}$$

and B_C is a Brownian bridge on $[0,1]^2$ with covariance function

$$\mathbb{E}\{B_C(u,v)\cdot B_C(u',v')\} = C(u\wedge u', v\wedge v') - C(u,v)\cdot C(u',v'),$$

with $u \wedge u' = \min(u, u')$ *and* C *the copula function with respect to (2.1).*

From (2.4), we introduce the following estimator for the extremal coefficient

$$\widehat{\theta}_\eta = \frac{1 + 2\widehat{\nu}_\eta}{1 - 2\widehat{\nu}_\eta}. \tag{3.3}$$

Applying the delta method, the following corollary follows.

Corollary 2. *Under the assumption of Theorem 1, we have*

$$\sqrt{N}\left(\widehat{\theta}_\eta - \theta_\eta\right) \xrightarrow{d} (1 + \theta_\eta)^2 \int_{[0,1]^2} N_C(u, v) dJ(u, v).$$

To infer the value of η, we complement our method-of-moment via a Generalized Probability Weighted Moment (GPWM) approach (Diebolt *et al.*, 2008) based on the following moment equality

$$\mu_\omega = \mathbb{E}(M^* \omega(F_{M^*})) = \int_{-\infty}^{+\infty} x\, \omega(F_{M^*}(x)) dF_{M^*}(x),$$

for any variable M^* with a distribution function F_{M^*} and ω a suitable continuous function satisfying

$$\begin{cases} \omega(t) = O((1-t)^b) & \text{for } t \text{ close to } 1,\ b \geq 0, \\ \omega(t) = O(t^{a'}) & \text{for } t \text{ close to } 0,\ a' > 0. \end{cases} \tag{3.4}$$

If we take $M^* = \max(M^*_{X,n}, M^*_{Y,n})$, whose distribution function equals $F_{M^*}(x) = \exp\{-V_\eta(1,1)x^{-\frac{1}{\eta}}\}$ (using Equation (2.1) and the homogeneous property of V_η) and if $\omega(t) := \omega_{a,b}(t) = t^a(-\log t)^b, a > a'$ then Diebolt *et al.* (2008) proved that

$$\mu_{a,b} := \mu_\omega = \frac{V_\eta^\eta(1,1)}{(a+1)^{b-\eta+1}} \Gamma(b - \eta + 1) \tag{3.5}$$

where $\Gamma(\alpha) = \int_0^\infty x^{\alpha-1} e^{-x} dx$.
A natural estimator for $\mu_{a,b}$ can be obtained by replacing F_{M^*} by its empirical version $\mathbb{F}_n$

$$\widehat{\mu}_{a,b} = \int_0^1 \mathbb{F}_n^{-1}(u) u^a (-\log u)^b du.$$

Using (3.5) with suitable values for (a, b) allows us to obtain an estimator for η in function of $\widehat{\mu}_{a,b}$

$$\widehat{\eta}_{gpwm} = 2\left(1 - \frac{\widehat{\mu}_{1,2}}{\widehat{\mu}_{1,1}}\right). \tag{3.6}$$

The asymptotic normality of $\widehat{\eta}_{gpwm}$ can then be deduced from the asymptotic properties of the GPWM estimator, see our Appendix.

Proposition 3. *Let* $\left(M^*_{X_i,n}, M^*_{Y_i,n}\right)$ *be a sample of* N *bivariate maxima vectors of block size* n *that follows a bivariate extreme value distribution with an* $\eta-$*dependence function defined as in (1.6). Then the GPWM estimator of* η *defined by* $\widehat{\eta}_{gpwm}$ *converges in the following way*

$$\sqrt{N}\left(\widehat{\eta}_{gpwm} - \eta\right) \xrightarrow{d} \frac{\eta 2^{3-\eta}}{\Gamma(2-\eta)} \left[I_1 - (1-\eta/2) I_2\right],$$

with $I_1 = \int_0^1 B(t)(-\log t)^{-\eta+1}\, dt$, $I_2 = \int_0^1 B(t)(-\log t)^{-\eta}\, dt$ *and* B *a Brownian bridge.*

3.2. The maximum likelihood approach

Besides our aforementioned method-of-moment approach, a Maximum Likelihood (ML) method can also be implemented. Our ML method is based on the normalized sample $\{M_i\} = \left\{\max\left(\frac{M^*_{X_i,n}}{\sigma^\eta_X}, \frac{M^*_{Y_i,n}}{\sigma^\eta_Y}\right)\right\}$, $i = 1, ..., N$ which admits the following log-likelihood

$$\log L(M_1, ..., M_N; \theta_\eta, \eta) = N \log\left(\frac{\theta_\eta}{\eta}\right) - (1/\eta + 1) \sum_{i=1}^{N} \log(M_i) - \theta_\eta \sum_{i=1}^{N} M_i^{-1/\eta}.$$

If $\widehat{\eta}_{mle}$ denotes the ML estimator for η based on the univariate sample $\left\{\max\left(M^*_{X_i,n}, M^*_{Y_i,n}\right)\right\}$ with a distribution function given by (2.1), it allows us to derive

$$\widehat{\theta}_{\eta,mle} = \left[\frac{1}{N}\sum_{i=1}^{N} \min\left(\widehat{\sigma}_{X,mle}(M^*_{X_i,n})^{-1/\widehat{\eta}_{mle}}, \widehat{\sigma}_{Y,mle}(M^*_{Y_i,n})^{-1/\widehat{\eta}_{mle}}\right)\right]^{-1}.$$

The estimates for σ_X and σ_Y in the above equality can be derived from (2.2) as $\widehat{\sigma}_{X,mle} = \left[\frac{1}{N}\sum_{i=1}^{N}(M^*_{X_i,n})^{-1/\widehat{\eta}_{mle}}\right]^{-1}$ and a similar expression for $\widehat{\sigma}_{Y,mle}$. Thus we can define

(3.7)

$$\widehat{\theta}_{\eta,mle} = \left[\frac{1}{N}\sum_{i=1}^{N} \min\left(\frac{(M^*_{X_i,n})^{-1/\widehat{\eta}_{mle}}}{\frac{1}{N}\sum_{j=1}^{N}(M^*_{X_j,n})^{-1/\widehat{\eta}_{mle}}}, \frac{(M^*_{Y_i,n})^{-1/\widehat{\eta}_{mle}}}{\frac{1}{N}\sum_{j=1}^{N}(M^*_{Y_j,n})^{-1/\widehat{\eta}_{mle}}}\right)\right]^{-1}.$$

4. EXAMPLES

4.1. A small simulation

To compare our estimators with the ML approach, we simulate 300 samples of 500 pairs of maxima from the η-asymmetric logistic dependence model (see Ramos and Ledford, 2011):

$$V_\eta(x, y) = \frac{1}{2 - 2^{\alpha/\eta}}(x^{-1/\alpha} + y^{-1/\alpha})^{\alpha/\eta}, \text{ for } x, y > 0,$$

with $\alpha \in \{0.1, 0.3, 0.5, 0.6\}$ and $\eta = 0.7$. This specific value of η corresponds to a case of asymptotic independence ($\eta < 1$) and provides $\theta_\eta = 2^{\alpha/\eta}$.

Boxplots of the estimators of θ and η are given in Figures 2 and 3 for different values of α and η. In these figures, the red square represents the true value of the parameter whereas the horizontal line is the median based on the 300 simulations.

In Figure 2 we can observe that the estimate $\widehat{\theta}_{\eta,mle}$ from (3.7) has a higher variability than $\widehat{\theta}_\eta$ from (3.3). This is particularly true when α is close to η, i.e. $\widehat{\theta}_\eta$ near two.
Concerning the estimation of η, Figure 3 basically tells the opposite story. The ML approach appears slightly better than the method-of-moment. This small simulation study advocates for not restricting one inference approach but rather to combine or at least compare different inference techniques.

4.2. Orgeval Rainfall data

Table 1 summarizes our inference with respect to the maxima plotted in Figure 1.

If one has to guess from Figure 1 some information about the degree of dependence, precipitation maxima during the Summer season clearly appear to be the less uncorrelated, followed by the Winter ones. The Spring and Fall seasons seem to witness a stronger and similar dependence.

Concerning the GPWM approach, from Table 1 we can see that the Spring and Fall seasons basically have the same η and the same θ. This parallel confirms Figure 1 where the points are strongly clustered for those two panels. Concerning the Winter and Summer seasons, the corresponding $\widehat{\theta}_\eta$ are much alike, but the $\widehat{\eta}_{gpwm}$ are different. Visually, this does not contradict the Winter and Summer displays, but it is not straightforward to interpret such results.

From Table 1, $\widehat{\eta}_{mle}$ appears to be almost equal to 0.7 for all seasons, but the Fall. It is puzzling that the Spring season belongs to this group because Figure 1 and the GWPM approach clearly discriminates the Spring from the Winter and Summer seasons. On the positive for the MLE approach, having the same η for the Winter, Spring and Summer, we can compare the ML estimates of θ. The ordering among those three $\widehat{\theta}_{\eta,mle}$ is coherent with Figure 1, the Summer has the largest value and the Spring the smallest one. The Fall season is difficult to interpret with the MLE approach, $\widehat{\eta}_{mle}$ being quite different to the values in the other seasons.

Now, if we want to compare the two approaches, GPWM and MLE, looking at Table 1, we can see that $\widehat{\theta}_\eta$ is quite stable, which is not the case for $\widehat{\theta}_{\eta,mle}$. At first sight, as both quantities estimate θ, it is difficult to know what to conclude. However, if we look at Figure 2(b) where the value of θ_η is in the range 1.3-1.4 (corresponding to the values given in Table 1) we can observe that indeed the variability with the maximum likelihood approach is more important than with the moment method. Thus this corroborates the instability of $\widehat{\theta}_{\eta,mle}$ observed in Table 1.

Overall, the time period of 1987-2002 may be too short to clearly compare the dependence among different seasons. Still, this example illustrates that analyzing jointly θ and η can bring relevant information that may not be obtained by simply interpreting θ.

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APPENDIX

Proof of (2.4). Applying the equality $|a-b|/2 = \max(a,b) - (a+b)/2$ to ν_η, we get:

$$\frac{1}{2}\mathbb{E}\,|F_X^*(M_X^*) - F_Y^*(M_Y^*)| = \mathbb{E}\max\{F_X^*(M_X^*), F_Y^*(M_Y^*)\} - \frac{1}{2}.$$

Then we calculate

$$\begin{aligned}\mathbb{P}[\max\{F_X^*(M_X^*), F_Y^*(M_Y^*)\} \leq u] &= \mathbb{P}[M_X^* \leq F_X^{*\leftarrow}(u), M_Y^* \leq F_Y^{*\leftarrow}(u)] \\ &= \exp\{-V_\eta(F_X^{*\leftarrow}(u), F_Y^{*\leftarrow}(u))\} \\ &= \exp\{\log(u) V_\eta(\sigma_X^\eta, \sigma_Y^\eta)\} = u^{V_\eta(\sigma_X^\eta, \sigma_Y^\eta)}\end{aligned}$$

because from the margin model (2.2) we have $F_X^{*\leftarrow}(u) = (-\log(u)/\sigma_X)^{-\eta}$ and $F_Y^{*\leftarrow}(u) = (-\log(u)/\sigma_Y)^{-\eta}$. Therefore, $\mathbb{E}\max\{F_X^*(M_X^*), F_Y^*(M_Y^*)\} = \frac{V_\eta(\sigma_X^\eta,\sigma_Y^\eta)}{1+V_\eta(\sigma_X^\eta,\sigma_Y^\eta)}$ from which (2.4) follows.

Proof of Theorem 1. First, we introduce the 'normalized' empirical distribution functions

$$\widetilde{F}_{n,N,X}(u) := \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\left[b_n^{-1}M^*_{X_i,n}\le u\right]}, \quad \widetilde{F}_{n,N,Y}(u) := \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\left[b_n^{-1}M^*_{Y_i,n}\le u\right]},$$

and we rewrite the estimator of the madogram as

$$\widehat{\nu}_\eta = \frac{1}{2N}\sum_{i=1}^{N}\left|\widetilde{F}_{n,N,X}\left(b_n^{-1}M^*_{X_i,n}\right) - \widetilde{F}_{n,N,Y}\left(b_n^{-1}M^*_{Y_i,n}\right)\right|.$$

Before starting the proof, we need to introduce a series of definitions linked to the copula function C. Although very similar, these definitions represent slightly different estimators of the same copula function. One difficulty of the proof is to show how close these versions are:

$$\widetilde{C}_{n,N}(u,v) := \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\left\{\widetilde{F}_{n,N,X}\left(b_n^{-1}M^*_{X_i,n}\right)\le u, \widetilde{F}_{n,N,Y}\left(b_n^{-1}M^*_{Y_i,n}\right)\le v\right\}},$$

$$C_{n,N}(u,v) := \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\left\{b_n^{-1}M^*_{X_i,n}\le \widetilde{F}^{\leftarrow}_{n,N,X}(u), b_n^{-1}M^*_{Y_i,n}\le \widetilde{F}^{\leftarrow}_{n,N,Y}(v)\right\}},$$

$$\widetilde{C}^*_{n,N}(u,v) := \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}_{\left\{U_{X_i,n}\le \widetilde{F}^{*\leftarrow}_{n,N,X}(u), V_{Y_i,n}\le \widetilde{F}^{*\leftarrow}_{n,N,Y}(v)\right\}}$$

where

$$\begin{aligned}
\widetilde{F}_{X,n}(z) &:= \mathbb{P}\left(b_n^{-1}M^*_{X_i,n}\le z\right), \quad \widetilde{F}_{Y,n}(z) := \mathbb{P}\left(b_n^{-1}M^*_{Y_i,n}\le z\right),\\
U_{X_i,n} &:= \widetilde{F}_{X,n}\left(b_n^{-1}M^*_{X_i,n}\right), \quad V_{Y_i,n} := \widetilde{F}_{Y,n}\left(b_n^{-1}M^*_{Y_i,n}\right),\\
\widetilde{F}^*_{n,N,X}(u) &:= \frac{1}{N}\sum_{i=1}^{N}\mathbb{1}_{\{U_{X_i,n}\le u\}}, \quad \widetilde{F}^*_{n,N,Y}(v) := \frac{1}{N}\sum_{i=1}^{N}\mathbb{1}_{\{V_{Y_i,n}\le v\}}.
\end{aligned}$$

The proof of Theorem 1 is divided into the following five steps.
Step 1. The function $\widetilde{C}_{n,N}(u,v)$ is very similar to $C_{n,N}(u,v)$, i.e. $\sup_{0\le u,v\le 1}|\widetilde{C}_{n,N}(u,v) - C_{n,N}(u,v)| \le 2/N$.
Step 2. We have $C_{n,N}(u,v) = \widetilde{C}^*_{n,N}(u,v)$.
Step 3. Define now the empirical distribution function of $(U_{X_i,n}, V_{Y_i,n})$ as

$$\widetilde{H}^*_{n,N}(u,v) = \frac{1}{N}\sum_{i=1}^{N}\mathbb{1}_{\left\{U_{X_i,n}\le u, V_{Y_i,n}\le v\right\}}$$

and its non-empirical version as

$$\widetilde{H}^*_n(u,v) = \mathbb{P}\left(\widetilde{F}_{X,n}\left(b_n^{-1}M^*_{X_i,n}\right)\le u, \widetilde{F}_{Y,n}\left(b_n^{-1}M^*_{Y_i,n}\right)\le v\right).$$

We establish that the process $\sqrt{N}(\widetilde{H}^*_{n,N} - \widetilde{H}^*_n)$ tends in distribution to a Brownian bridge B_C. To this end, we prove the convergence of the finite-dimensional distributions and the tightness of the process.
Step 4. The process $\sqrt{N}(\widetilde{C}^*_{n,N} - \widetilde{H}^*_n)$ tends in distribution to a Gaussian process N_C.
Step 5. We conclude the proof of our theorem using the integration by parts.
This proof is only sketched here as it is a slightly modified version of the one of Proposition 4 in Naveau *et al.* (2009) which is detailed in
http://sama.ipsl.jussieu.fr/Documents/articles/NaveauBiometrika07DetailedProofs.pdf.

Remark about our Theorem 1. The limiting process is such that

$$\int_{[0,1]^2} N_C(u,v)dJ(u,v) = \frac{1}{2}\int_0^1 N_C(0,v)dv + \frac{1}{2}\int_0^1 N_C(u,0)du - \int_0^1 N_C(u,u)du.$$

This limiting process cannot be precised without specifying the copula function and in special cases where these integrals can be computed. For instance, consider the Product copula, also called the independent copula, defined as $C(u,v) = uv$. In that case

$$N_C(u,v) = B_C(u,v) - vB_C(u,1) - uB_C(1,v),$$

from which direct computations lead to

$$\mathrm{Var}\left(\int_{[0,1]^2} N_C(u,v)dJ(u,v)\right) = \frac{1}{90}.$$

Proof of Proposition 3. As $\eta \in (0,1]$, we have according to Theorem 2.1 in Diebolt *et al.* (2008) that
(4.1)

$$\sqrt{N}\begin{pmatrix} \widehat{\mu}_{1,1} - \mu_{1,1} \\ \widehat{\mu}_{1,2} - \mu_{1,2} \end{pmatrix} \xrightarrow{d} \begin{pmatrix} \eta V_\eta^\eta(1,1) \int_0^1 \frac{B(t)}{t}\Big(-\log t\Big)^{-\eta-1} t(-\log t)dt \\ \eta V_\eta^\eta(1,1) \int_0^1 \frac{B(t)}{t}\Big(-\log t\Big)^{-\eta-1} t(-\log t)^2 dt \end{pmatrix},$$

where B denotes a Brownian bridge and $n \to \infty$. It follows

$$\begin{aligned} \sqrt{N}\left(\widehat{\eta}_{gpwm} - \eta\right) &= -\frac{2}{\widehat{\mu}_{1,1}\mu_{1,1}}\sqrt{N}\left(\mu_{1,1}\widehat{\mu}_{1,2} - \mu_{1,2}\widehat{\mu}_{1,1}\right) \\ &= -\frac{2}{\widehat{\mu}_{1,1}\mu_{1,1}}\left[\mu_{1,1}\sqrt{N}(\widehat{\mu}_{1,2} - \mu_{1,2}) - \mu_{1,2}\sqrt{N}(\widehat{\mu}_{1,1} - \mu_{1,1})\right]. \end{aligned}$$

An application of Slutzky's theorem leads to

$$\sqrt{N}\left(\widehat{\eta}_{gpwm} - \eta\right) \xrightarrow{d} -\frac{2\eta V_\eta^\eta(1,1)}{\mu_{1,1}^2}\int_0^1 \frac{B(t)}{t}\Big(-\log t\Big)^{-\eta-1}\left[\mu_{1,1}\omega_{1,2}(t) - \mu_{1,2}\omega_{1,1}(t)\right]dt,$$

from which Proposition 3 follows.

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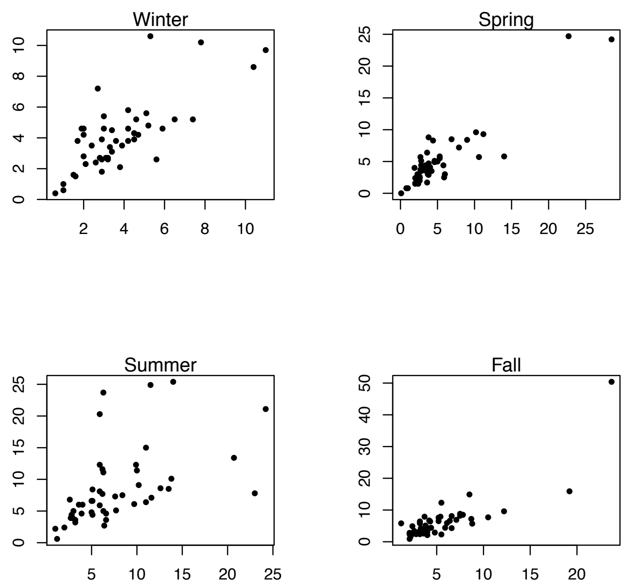
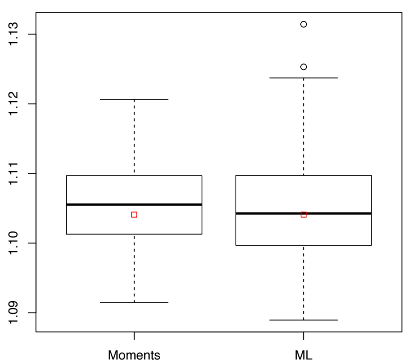


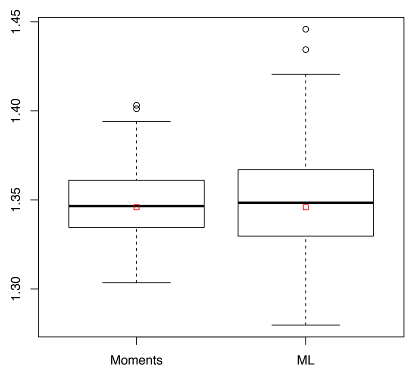
Figure 1: Monthly maxima of hourly precipitation for each season, measured at two stations in the basin of Orgeval (near Paris) during 1987-2002.

	$\widehat{\theta}_\eta$	$\widehat{\theta}_{\eta,mle}$	$\widehat{\eta}_{gpwm}$	$\widehat{\eta}_{mle}$
winter	1.44	1.26	0.44	0.71
spring	1.33	1.22	0.50	0.70
summer	1.45	1.47	0.56	0.72
fall	1.36	1.60	0.49	0.51

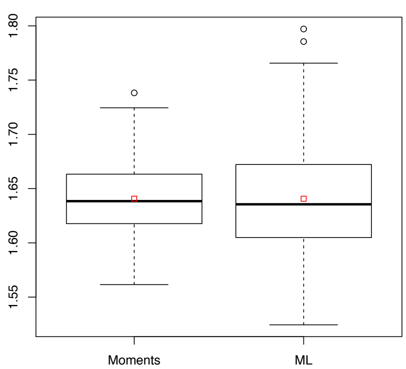
Table 1: Estimates with GPWM and ML approaches for the Orgeval rainfall data



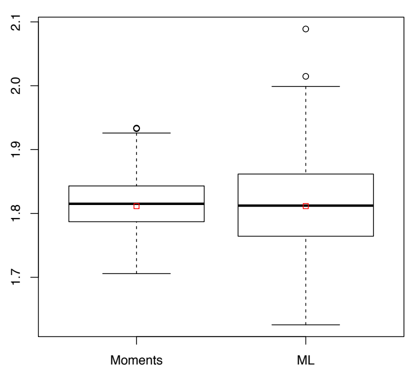
(a) Case $\alpha = 0.1, \eta = 0.7$



(b) Case $\alpha = 0.3, \eta = 0.7$

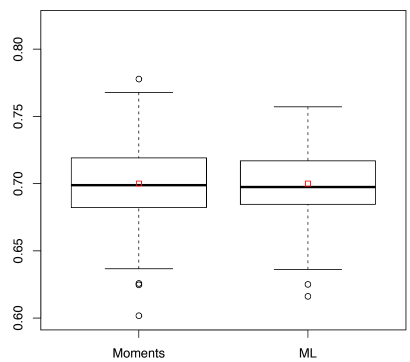


(c) Case $\alpha = 0.5, \eta = 0.7$

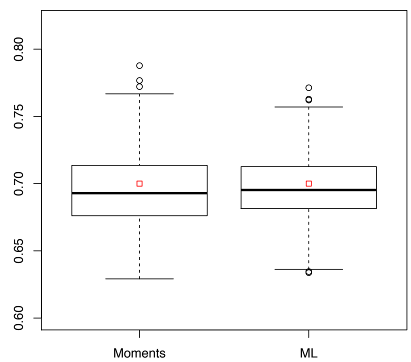


(d) Case $\alpha = 0.6, \eta = 0.7$

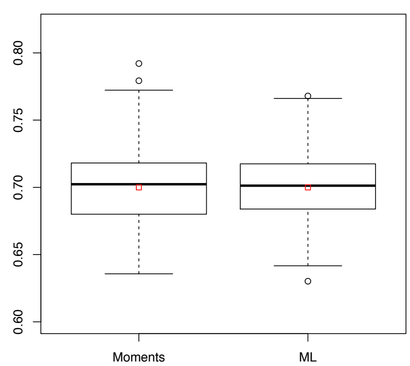
Figure 2: Simulation: comparing $\widehat{\theta}_\eta$ from (3.3) with $\widehat{\theta}_{\eta,mle}$ from (3.7)



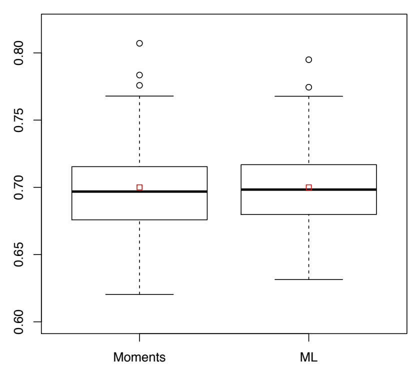
(a) Case $\alpha = 0.1, \eta = 0.7$



(b) Case $\alpha = 0.3, \eta = 0.7$



(c) Case $\alpha = 0.5, \eta = 0.7$



(d) Case $\alpha = 0.6, \eta = 0.7$

Figure 3: Simulation: comparing $\widehat{\eta}_{gpwm}$ from (3.6) with $\widehat{\eta}_{mle}$.