A new parametric class of cross-covariance functions

For multivariate spatio-temporal random fields

Marc Bourotte[1], Denis Allard[1] & Emilio Porcu[2]

[1] INRA Avignon - Unité BioSP
[2] University Federico Santa Maria, Chile

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- Use of Gaussian Random Field (latent or not) to model climatic variables
- Multivariate space-time data

$\Rightarrow$ **Need to model a multivariate spatio-temporal second order structure**

Separability

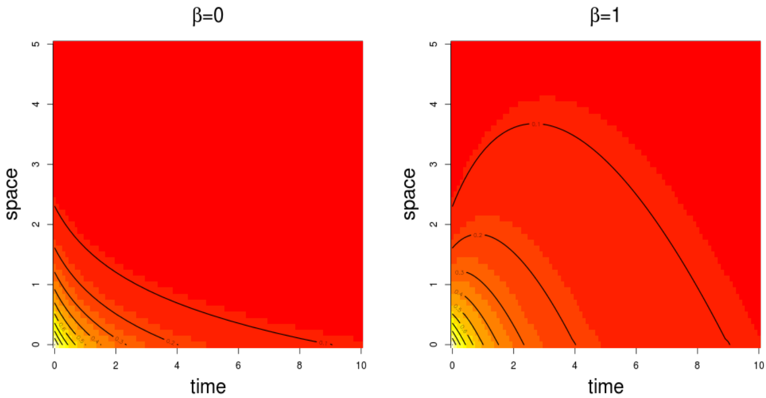
$M \bigotimes S \bigotimes T$: For all $(\mathbf{h}, u)$ in $\mathbb{R}^d \times \mathbb{R}$ and all $i, j = 1, ..., p$

$$C_{ij}(\mathbf{h}, u) = \mathbb{C}ov\left(Z_i(s,t), Z_j(s+h, t+u)\right) = \rho_{ij}.C_S(\mathbf{h}).C_T(u)$$

Nonseparable models :

- $M \bigotimes (ST)$: $C_{ij}(\mathbf{h}, u) = \rho_{ij}.C(\mathbf{h}, u)$
- $(MS) \bigotimes T$: $C_{ij}(\mathbf{h}, u) = C_{ij}(\mathbf{h}).C_T(u)$

New nonseparable model for multivariate spatio-temporal random fields (MST)



$$C(\mathbf{h}, u|\beta) = (|u| + 1)^{-1} \exp\left(-\frac{\|\mathbf{h}\|}{(|u|+1)^{\frac{\beta}{2}}}\right)$$

Outline

1. Existing models & Theorems
2. A new cross-covariance model for multivariate space-time data
3. Simulations
4. Application to data
5. Conclusion

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Nonseparable space-time covariance function (*Gneiting, 2002*)

If

- $\varphi(x), x \geqslant 0$, a completely monotone function
- $\psi(x), x \geqslant 0$, a positive function with completely monotone derivative

then

$$C(\mathbf{h}, u) = \frac{\sigma^2}{\psi(|u|^2)^{\frac{d}{2}}}\, \varphi\left(\frac{||\mathbf{h}||^2}{\psi(|u|^2)}\right), \quad (\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{R} \qquad (1)$$

is a valid space-time covariance function.

$$\varphi(x) = \exp(-rx^{\gamma}) \qquad r > 0,\ 0 < \gamma \leqslant 1$$

$$\varphi(x) = \frac{2^{1-\nu}}{\Gamma(\nu)}(rx^{\frac{1}{2}})^{\nu}\mathcal{K}_{\nu}(cx^{\frac{1}{2}}) \qquad c > 0,\ \nu > 0$$

$$\psi(x) = (ax^{\alpha} + 1)^{\beta} \qquad a > 0,\ 0 < \alpha \leqslant 1,\ 0 \leqslant \beta \leqslant 1$$

Matérn cross-covariance functions for multivariate random fields (*Gneiting, Kleiber & Schlather, 2010*)

The matrix-valued covariance function $\mathbf{C}(\mathbf{h}) = [C_{ij}(\mathbf{h})]_{i,j=1}^{p}$ with $C_{ij}(\mathbf{h}) = \rho_{ij}\sigma_i\sigma_j M(\mathbf{h}|\nu_{ij}, r_{ij})$ define a valid second-order structure if

- $$\nu_{ij} = \frac{1}{2}(\nu_i + \nu_j) \tag{2}$$
- $$r_1 = ... = r_p = r \text{ and } r_{ij} = r > 0 \tag{3}$$
- $$\rho_{ij} = \beta_{ij} \frac{\Gamma(\nu_i + \frac{d}{2})^{\frac{1}{2}}}{\Gamma(\nu_i)^{\frac{1}{2}}} \frac{\Gamma(\nu_j + \frac{d}{2})^{\frac{1}{2}}}{\Gamma(\nu_j)^{\frac{1}{2}}} \frac{\Gamma\left(\frac{1}{2}(\nu_i + \nu_j)\right)}{\Gamma\left(\frac{1}{2}(\nu_i + \nu_j + d)\right)} \tag{4}$$

where the matrix $[\beta_{ij}]_{i,j=1}^{p}$ is a correlation matrix.

Each marginal function is of the Matérn type.

$$C_{ii}(\mathbf{h}) = \sigma_i^2 \frac{2^{1-\nu_i}}{\Gamma(\nu_i)} (r\|\mathbf{h}\|)^{\nu_i} \mathcal{K}_{\nu_i}(r\|\mathbf{h}\|) = M(\mathbf{h}|\nu_i, r)$$

Bernstein's theorem (*Feller, 1966*)

A completely monotone function $\varphi(t), t \geqslant 0$, can be written as

$$\varphi(t) = \int_0^\infty \exp(-rt)dF(r) \tag{5}$$

where F is nondecreasing.

Theorem (*Schlather,2010*)

A matrix-valued covariance function $\mathbf{C}(\mathbf{h}, u) = [C_{ij}(\mathbf{h}, u)]_{i,j=1}^p$ defined on $\mathbb{R}^d \times \mathbb{R}$ is valid if and only if

$$\left[C_{ij,\mathbf{w}}(u) := \int e^{-i\mathbf{h}'\mathbf{w}} C_{ij}(\mathbf{h}, u)d\mathbf{h}\right]_{i,j=1}^p \tag{6}$$

is positive definite for almost all $\mathbf{w} \in \mathbb{R}^d$.

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Proposition

The following matrix of covariance functions $\mathbf{C}(\mathbf{h}, u) = [C_{ij}(\mathbf{h}, u)]_{i,j=1}^{p}$ with

$$C_{ij}(\mathbf{h}, u) = \frac{\sigma_i \sigma_j \rho_{ij}}{\psi(u^2)^{\frac{d}{2}}} M\left(\frac{\mathbf{h}}{\psi(u^2)^{\frac{1}{2}}} | \frac{1}{2}(\nu_i + \nu_j), r\right), \quad (\mathbf{h}, u) \in \mathbb{R}^d \times \mathbb{R} \qquad (7)$$

define a valid model for stationary multivariate spatio-temporal random field if

- the term ρ_{ij} is equal to

$$\rho_{ij} = \beta_{ij} \frac{\Gamma(\nu_i + \frac{d}{2})^{\frac{1}{2}}}{\Gamma(\nu_i)^{\frac{1}{2}}} \frac{\Gamma(\nu_j + \frac{d}{2})^{\frac{1}{2}}}{\Gamma(\nu_j)^{\frac{1}{2}}} \frac{\Gamma\left(\frac{1}{2}(\nu_i + \nu_j)\right)}{\Gamma\left(\frac{1}{2}(\nu_i + \nu_j + d)\right)}$$

- $\psi(t), t > 0$, is a positive function with completely monotone derivative
- $[\beta_{ij}]_{i,j=1}^{p}$ is a correlation matrix

Sketch of the proof : Combination of *GKS*, 2010 and *Schlather*, 2010

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The amount of data is equal to $n = p \times n_S \times n_T$.
Let $(Y_k, Y_{k'}), k, k' = 1, .., n,$ be a pair such as

$$\begin{aligned} Y_k &= Y_i(\mathbf{s}_j, t) \qquad i = 1, .., p \quad j = 1, .., n_S \quad t = 1, .., n_T \\ Y_{k'} &= Y_{i'}(\mathbf{s}_{j'}, t') \qquad i' = 1, .., p \quad j' = 1, .., n_S \quad t' = 1, .., n_T \end{aligned}$$

Weighted Pairwise (Log-)Likelihood

$$wpl(\theta) = \sum_{k=1}^{n-1} \sum_{k'=k+1}^{n} w_{kk'} \left\{ -\log(2\pi) - \frac{1}{2}\log|\Sigma_{kk'}| - \frac{1}{2}(Y_k, Y_{k'})\Sigma_{kk'}^{-1}(Y_k, Y_{k'})^T \right\} \tag{8}$$

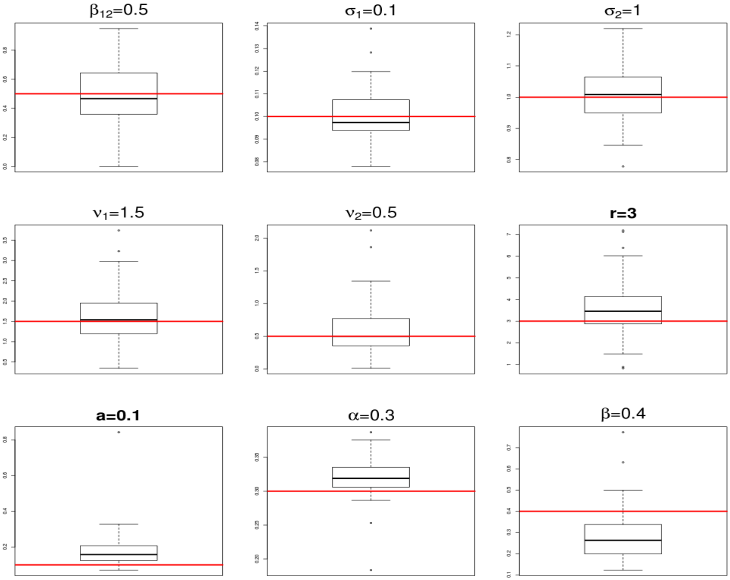
where

$$w_{kk'} = \begin{cases} 1 & \text{if} \quad \mathbf{h} = \|\mathbf{s}_j - \mathbf{s}_{j'}\| \leqslant \lim_S \text{ and } \quad u = |t - t'| \leqslant \lim_T \\ 0 & \text{otherwise} \end{cases}$$

$$\Sigma_{kk'} = \begin{pmatrix} \sigma_i^2 & C_{ii'}(\mathbf{h}, u) \\ C_{ii'}(\mathbf{h}, u) & \sigma_{i'}^2 \end{pmatrix}$$

Parameters to estimate

Parameters	Number	Initial Values
σ_i	p	empirical standard deviation
β_{ij}	$\frac{p(p-1)}{2}$	empirical correlation
ν_i, r	$p+1$	$C_{ii}(\mathbf{h},0) = \sigma_i^2 M(\mathbf{h}\|\nu_i, r)$
a, α, β	3	$C_{ii}(\mathbf{0},u) = \sigma_i^2 (au^{2\alpha}+1)^{-\frac{\beta d}{2}}$


β_{12}=0.5
σ_1=0.1
σ_2=1
ν_1=1.5
ν_2=0.5
r=3
a=0.1
α=0.3
β=0.4

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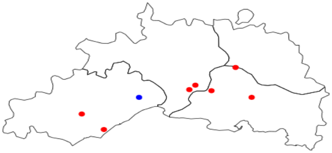


Data from portal Climatik INRA : 2 climatic variables observed at 8 weather stations during 11 years.

- Daily Mean Temperature (°C)
- Daily Mean Humidity (%)
- 11 Winters

Goal : Conditional simulation of mean temperature and humidity at Montpellier (blue point) for winter 2011-2012.

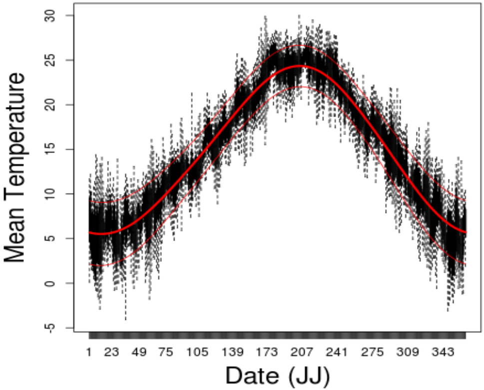


Figure: Mean temperature at Avignon 2001-2012. Bold red line : seasonal mean ajusted by splines.

Temperature & Humidity model

$$\mathbf{Y} = (Y_1(s_1,t_1), Y_2(s_1,t_1), \ldots, Y_2(s_8,t_1), Y_1(s_1,t_2), \ldots, Y_2(s_8,t_n))^T$$

$$\mathbf{Y} = \text{seasonal trend} + \text{error} \quad \text{where the error} \sim \mathcal{N}(0,\Sigma)$$

Correlation model and parameters estimated

$$C_{12}(\mathbf{h},u) = \frac{\beta_{12}}{a|u|^{2\alpha}+1} \frac{\Gamma(\nu_1+\frac{d}{2})^{\frac{1}{2}}}{\Gamma(\nu_1)^{\frac{1}{2}}} \frac{\Gamma(\nu_2+\frac{d}{2})^{\frac{1}{2}}}{\Gamma(\nu_2)^{\frac{1}{2}}} \frac{\Gamma\left(\frac{1}{2}(\nu_1+\nu_2)\right)}{\Gamma\left(\frac{1}{2}(\nu_1+\nu_2+d)\right)} M\left(\frac{\mathbf{h}}{(a|u|^{2\alpha}+1)^{\frac{\beta}{2}}} | \frac{1}{2}(\nu_1+\nu_2), r\right)$$

$$\beta_{12} = 0.41 \quad \nu_1 = 0.48 \quad \nu_2 = 0.30 \quad r = 1.35 * 10^{-3} \quad (1/r = 740km)$$

$$a = 0.45 \quad \alpha = 0.71 \quad \beta = 0.55 \quad (d = 2 \text{ because } \mathbf{h} \in \mathbb{R}^2)$$

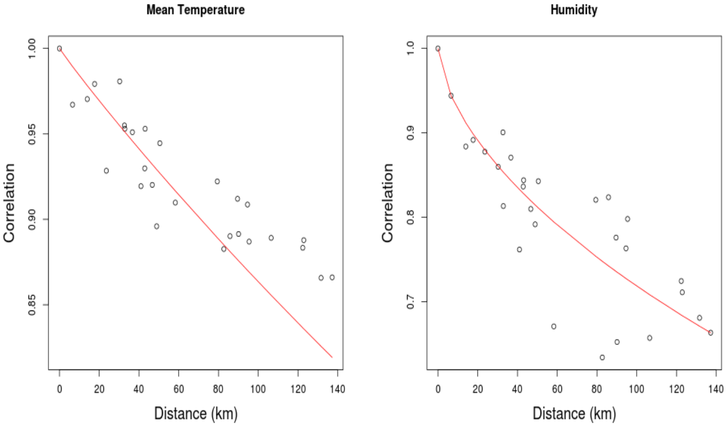


Figure: Marginal spatial correlation for each variable ($u = 0$).

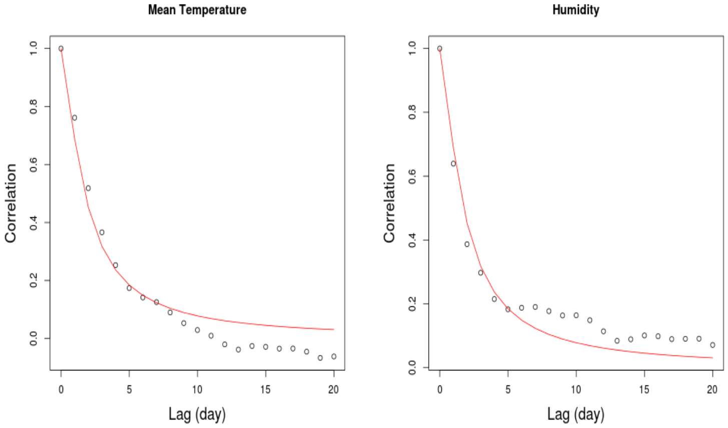
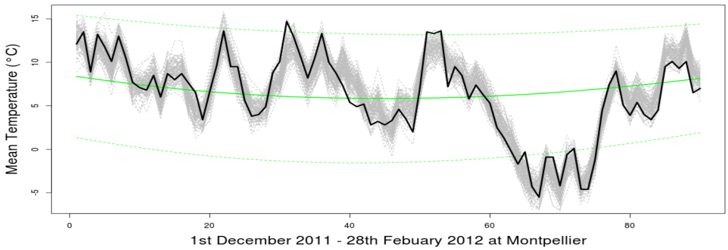
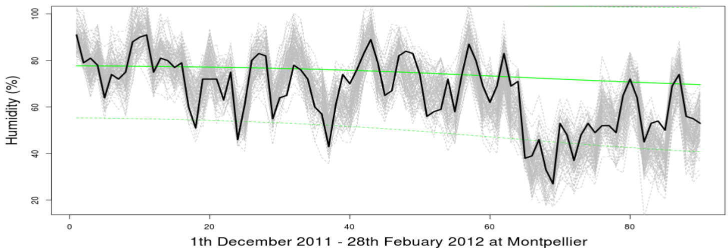


Figure: Marginal temporal correlation for each variable ($h = 0$).


Mean Temperature (°C)
-5
0
5
10
15
0
20
40
60
80
1st December 2011 - 28th Febuary 2012 at Montpellier


Humidity (%)
20
40
60
80
100
0
20
40
60
80
1th December 2011 - 28th Febuary 2012 at Montpellier

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CONCLUSION

- New nonseperable cross-covariance model for multivariate space-time stationary random field
  - Theorical validity
  - Simulation and estimation procedures

- First application : gives coherent results, need to be compared with other methods

- Limitation : symetric model

- Future work :
  - Theorical work : $\psi(u) = (a|u|^{\alpha} + 1)^{\beta} \rightarrow \psi_{ij}(u) = (a_{ij}|u|^{\alpha_{ij}} + 1)^{\beta_{ij}}$
  - Data application : more stations into a larger area // more climatic variables