Meta-elliptical extremes in finite and infinite dimensions

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PEPER Workshop, Aussois

17/12/2013

Objectives

- characterize the limiting behavior of asymptotically dependent elliptical distributions ;
- construct the corresponding **max-stable** limit distributions (**extremal-**t **limits**) ;
- generalize to **infinite dimensions** (spatial domain) ;
- present a method for **efficient, exceedance-based likelihood** inference.

The framework of asymptotic dependence

- Random variables X, Y are **asymptotically dependent** if

$$\lambda(X,Y) = \lim_{u \uparrow 1} \mathrm{pr}(F_X(X) \geq u \mid F_Y(Y) \geq u) > 0.$$

- In the spatial domain :

$$\lambda(X(s_1), X(s_2)) > 0$$

The frequency of very extreme co-occurrences is of the same order of magnitude as that of very extreme single events.

Plan

Regular variation and ellipticity

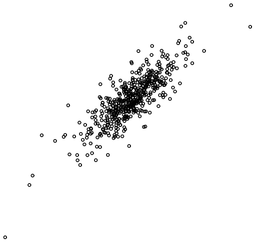
The max-stable limit process

Outlook : full likelihood inference with partial censoring

Conclusion

Elliptical distributions

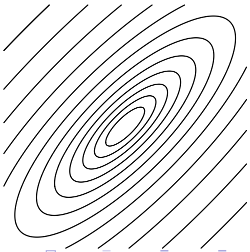
[Cambanis et al., 1981, Anderson and Fang, 1990], ...



Stochastic polar representation $\mathbf{X} \overset{d}{=} RA\mathbf{U} + \mathbf{M}$

with

- a random radius $R \geq 0$;
- a covariance matrix $\Sigma = AA^T$,
  assumed to be invertible in the following ;
- a random vector $\mathbf{U}$ uniform on $\{\mathbf{x} \mid \mathbf{x}^T\mathbf{x} = 1\}$,
  independent of R ;
- a median vector $\mathbf{M}$;
- spatial : elliptical random field.



Gaussian-based examples

- Gaussian $\mathbf{G}$: asymptotic **in**dependence
- $R\mathbf{G}$ with Pareto-tailed $R \geq 0$: asymptotic dependence
  - e.g. student's t with $\mathrm{df} > 0$ degrees of freedom :
    $$(\mathrm{df}/R)^2 \sim \mathrm{Gamma}(\mathrm{df}/2, 2)$$
  - in the spatial domain : t **process**

Multivariate regular variation on $\mathbb{R}^d$

Univariate regular variation :

$$t\,\mathrm{pr}(X/a_t \geq x) \to x^{-\alpha}, \quad t \to \infty, \quad a_t \to \infty, \quad \text{with } \alpha > 0.$$

This corresponds to Pareto-like tail behavior.

Definition of multivariate regular variation

MRV of a random vector $\mathbf{X}$:

$\exists\ a_t \to \infty$ such that

$$\eta_t(\cdot) = t\,\mathrm{pr}(\mathbf{X}/a_t \in \cdot) \Longrightarrow \eta(\cdot) \quad \text{vaguely on } \overline{\mathbb{R}}^d \setminus \{\mathbf{0}\} \text{ for } t \to \infty.$$

- We can choose $a_t = F^{\leftarrow}_{\|\mathbf{X}\|}(1 - 1/t)$ for some norm $\|\cdot\|$.
- Homogeneity : $\eta(tB) = t^{-\alpha}\eta(B)$
  (**index of regular variation** $\alpha > 0$).

$\rightsquigarrow \mathrm{pr}(\mathbf{X} \in B) \approx c\eta(B)$ for "extreme" sets B separated far from $\mathbf{0}$.

[Hult and Lindskog, 2002] ; [Hashorva, 2006]

For $\mathbf{X} = RA\mathbf{U} + \mathbf{M}$,

asymptotic dependence

$\Leftrightarrow$

multivariate regular variation

$\Leftrightarrow$

R is univariate regularly varying

$\Leftrightarrow$

X_j is univariate regularly varying for all j

$\Leftrightarrow$

X_j is univariate regularly varying for some j

η is elliptically contoured, characterized by α and $\Sigma = AA^T$

Density representation of η

$$\eta(\mathrm{d}\mathbf{x}) = c_\alpha(\mathbf{x}^T\Sigma^{-1}\mathbf{x})^{-0.5(\alpha+d)}$$

with $c_\alpha = \alpha\pi^{-0.5(d-1)}|\Sigma|^{-0.5}\Gamma(0.5(\alpha+1))^{-1}\Gamma(0.5(\alpha+d))$.

Exhaustiveness of the extremal-t dependence

For the multivariate t distribution, we observe $\alpha = \mathrm{df}$

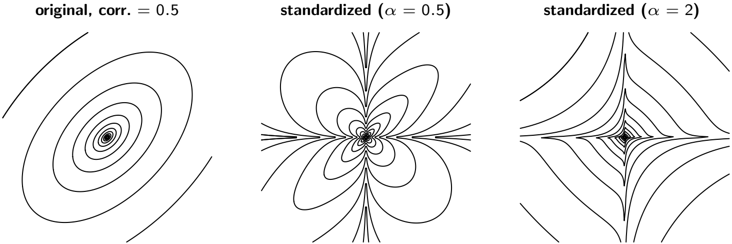
$\Rightarrow$ all asymptotically dependent limits are limits of t distributions.

Standardized marginal scale

Often we use a standardized version η^*,

$$\eta^*(B) = \eta(\mathbf{c}B^\alpha) \quad \text{with } \mathbf{c}B^\alpha = \{\mathbf{c} \cdot \text{sign}(\mathbf{x}) \cdot \text{abs}(\mathbf{x})^\alpha \mid \mathbf{x} \in B\}$$

such that $\eta^*(\{\mathbf{x} \mid x_j \geq x_0\}) = \eta^*(\{\mathbf{x} \mid x_j \leq -x_0\}) = x_0^{-1}$ for $x_0 > 0$.



α is now a concentration parameter of the tail dependence structure.
We can speak of the **meta-elliptical extreme value dependence**.

Max-stable limit processes

Talks of Aurélien Bechler, Raphaël Huser, Mathieu Ribatet, Marco Oesting, . . .

Definition : Maximum domain of attraction

Let be given i.i.d. copies $\mathbf{X}_\mathbf{i}$ of a stochastic process $\mathbf{X} = \{X(s)\}$.

- Max-domain of attraction :

$$\left\{ \max_{i=1,\ldots,n} a_n(s)^{-1} \left[X_i(s) - b_n(s)\right] \right\} \Longrightarrow \{Z(s)\} \quad (n \to \infty)$$

  in terms of **finite-dimensional distributions**.
  The limit process $\mathbf{Z} = \{Z(s)\}$ is max-stable.
- $X(s_1), X(s_2)$ asymptotically independent $\Leftrightarrow$ $Z(s_1), Z(s_2)$ independent.

- We can model blockwise maxima with a max-stable process $\mathbf{Z}$.
- Asymptotic considerations justify to model also threshold exceedances with max-stable processes.

The extremal-t max-stable limit process

- If any $X(s)$ of an elliptical random field is univariate regularly varying, then
  - all finite-dimensional distributions are MRV with some index α,
  - a dependent max-stable limit process exists.

  The limit process arises as the max-stable limit of random t fields.

- The limit process is characterized by a correlation function and the concentration parameter $\alpha > 0$.

Finite-dimensional distributions

We use the **standardized marginal scale.**

The cumulative distribution function for $s_1, \ldots, s_d$ is given as

$$\exp\left\{-V^*(z_1, \ldots, z_d)\right\}, \quad \mathbf{z} \in [0, \infty]^d,$$

with the dependence function

$$V^*(z_1, \ldots, z_d) = \eta^*_{s_1,\ldots,s_d}([-\infty, \mathbf{z}]^C).$$

Marginal distributions are unit Fréchet.

$V^*(z_1, \ldots, z_d)$ can be expressed in terms of multivariate t probabilities ([Nikoloulopoulos et al., 2009]).

Construction

Spectral constructions of max-stable processes are important for

- the definition and interpretation of **parametric models**,
- **simulation** and **conditional simulation**.

Extremal-*t* process [Opitz, 2013]

Let

- $\{V_i\}$ a unit rate Poisson process on $(0, \infty)$,
- $\mathbf{G_i}$ i.i.d. standard Gaussian with the targeted correlation function.

Then

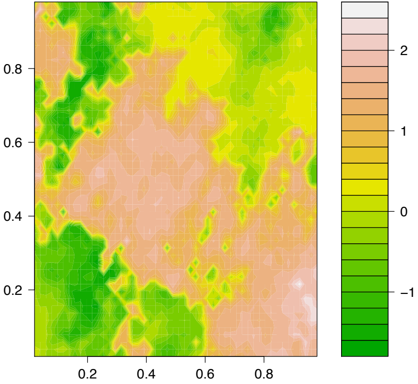
$$\mathbf{Z}^* = \{Z^*(s)\} = \left\{ \tilde{c}_\alpha \max_{i=1,2,\ldots} G_i(s)^\alpha / V_i \right\}$$

- The extremal Gaussian process [Schlather, 2002] is a special case ($\alpha = 1$).

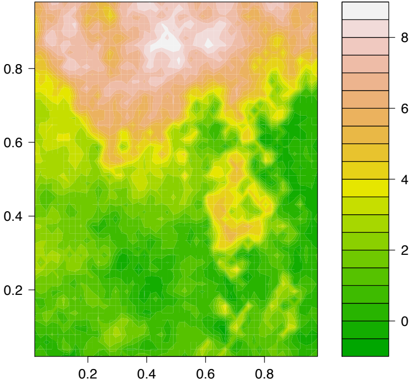
Examples of $\log(\mathbf{Z}^*)$

with exponential correlation

$\alpha = 0.5$
(stronger long-range dependence)



$\alpha = 5$
(weaker long-range dependence)



Inference

We focus on **inference based on threshold exceedances**. Typically, such estimation is more **efficient** than using maxima.

As a model, we can use either the max-stable process or the closely related Pareto process [Ferreira and de Haan, 2013, Dombry and Ribatet, 2013].

It can be useful to decouple the marginal behavior from the dependence structure.

Univariate extreme value theory suggests marginal tail parameters μ_j (position), $\sigma_j > 0$ (scale) and ξ_j (shape). The literature on their estimation is vast.

We focus on **estimation of the dependence structure** based on the standardized data

$$X_j^* = X^*(s_j) = 1/\left[1 - F_{X(s_j)}(X(s_j))\right], \quad j = 1, \ldots, d.$$

Likelihood inference with partial censoring of exceedances

Let $V^*(\mathbf{x}) = \eta^*_{s_1,\ldots,s_d}\left\{[\mathbf{0},\mathbf{x}]^C\right\}$ the dependence function.
Then $\Pr(\mathbf{X}^* \not\leq \mathbf{x}) \approx \mathrm{V}^*(\mathbf{x})$ for large $\mathbf{x}$.

Principle

- An event $\mathbf{X}^*$ is considered as extreme when a fixed threshold vector $\mathbf{u}$ is exceeded in at least one component, i.e. when $\max_j X_j^*/u_j \geq 1$.
- Components $X_j^* < u_j$ are censored.
- Likelihood contribution of $\mathbf{X}^*$:
  - when none of the components exceeds its thresholds : $1 - V^*(\mathbf{u})$;
  - when w.l.o.g. components $X_1^* = x_1, \ldots, X_{j_0}^* = x_{j_0}$ are exceedances :

$$-\frac{\partial^{j_0}}{\partial x_1 \times \ldots \times \partial x_{j_0}} V^*(x_1, \ldots, x_{j_0}, u_{j_0+1}, \ldots, u_d).$$

In general,

there are **two potential difficulties** with this approach :

- The dependence function V^* may be unknown for $d > 2$.
- If the dimension d is large, partial derivatives may become intractable.

Then only composite likelihood methods are possible.

Here :

- V^* is known.
- Partial derivates can be calculated for the extremal-t model even in large dimension (paper in preparation with Emeric Thibaud, EPF Lausanne).

Simulation results show **efficiency gains** due to the full likelihood approach.

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- Based on limits for asymptotically dependent elliptical processes, we get **flexible and interpretable models** for multivariate and spatial extremes.
- **Efficient likelihood inference** is possible.
- **Simulation** of the max-stable limits is possible through the presented construction, allowing for the numerical calculation of functionals $f(\mathbf{Z})$.

— Thank you for your attention —

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