

# Clustering of extremes for time series

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Peper Workshop

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# Outline

- 1 Motivation: realistic SCR calculation
- 2 Peaks Over Threshold approach
- 3 Precise large deviations in the iid case
- 4 Precise large deviations in the dependent case

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# Solvency II and the SCR

## Definition (Solvency Capital Requirement)

"The SCR is the capital required to ensure that the (re)insurance company will be able to meet its obligations over the next 12 months with a probability of at least 99.5%." (Wikipedia)

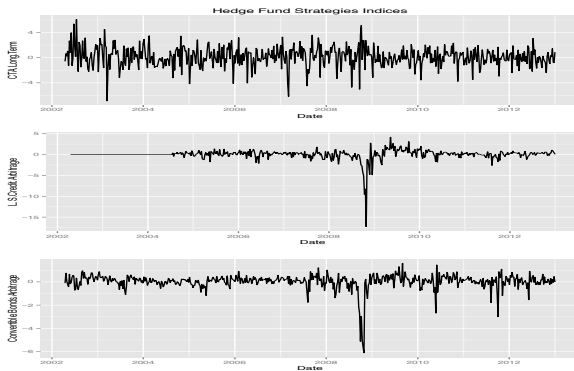
## Solvency II: a challenge for the mathematician

Use a standard formula ( $SCR=49\%$ ) or find a more realistic calculation of the SCR (quantiles, VaR) using an internal model.

Extrapolation of the magnitude of an event that occurs once per 200 years, i.e. that is not observed!

# Quantification of the dependence

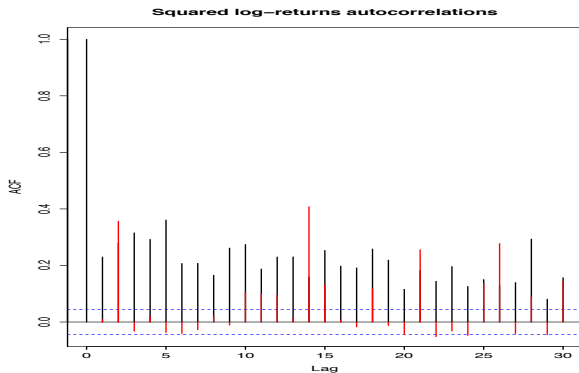
Log ratios  $r_t = \log(P_{t+1}/P_t)$  where  $(P_t)$  are weekly prices of HF indices.



3 different strategies with very different behaviors.

# Unstable estimations of the autocovariances

Assume  $(r_t)$  is strictly stationary. Squared of log ratios  $r_t^2$  have fat tails: estimations of the autocorrelations are unstable:



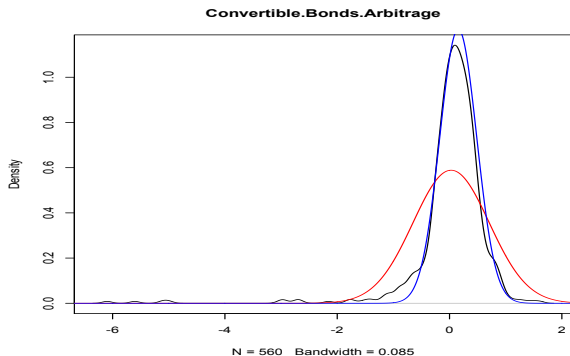
Mikosch & W. (2013)

It exists uncorrelated GARCH(1,1) models such that the autocorrelations converge but with errors that are non normal and totally skewed to the right...

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# Gaussian modeling works on average



Dark: marginal density of the log-ratio of an index

Red: Gaussian model

Blue: Gaussian model excluding the 40 worst days

# Gaussian modeling underestimate tails

$$\mathbb{P}\left(\frac{P_{T+1}^{an} - P_T}{P_T} \leq SCR\right) = 0.005 \quad \langle \approx \rangle \quad \mathbb{P}\left(\log\left(\frac{P_{T+1}^{an}}{P_T}\right) \leq SCR\right) = 0.005$$
$$\approx \mathbb{P}\left(\sum_{t=T}^{T+52} r_t \leq SCR\right) = 0.005.$$



Modern portfolio theory, Markowitz (1952)

# Peaks Over Threshold approach

To extrapolate, "Let the tails speak by themselves"! (EKM, 1997)

## Theorem (Pickands-Balkema-de Haan, 1975-1974)

Let  $(X_t)$  iid with distribution  $F$ . Denote  $F_u$  the distribution of the exceedances

$$F_u(x) = \mathbb{P}(X - u \leq x \mid X > u), \quad x \geq 0, \quad X \sim F.$$

Then for a large class of underlying distribution functions  $F$  with endpoint  $x_F$

$$\lim_{u \rightarrow x_F} \sup_{0 < x < x_F - u} |F_u(x) - G_{\xi, \sigma}(x)| = 0$$

where  $G_{\xi, \sigma}$  is the Generalized Pareto Distribution

$$G_{\xi, \sigma}(x) = \begin{cases} 1 - (1 - \xi x / \sigma)^{-1/\xi}, & \text{if } \xi \neq 0, \\ 1 - e^{-x/\sigma}, & \text{if } \xi = 0. \end{cases}$$

# Regularly varying distribution

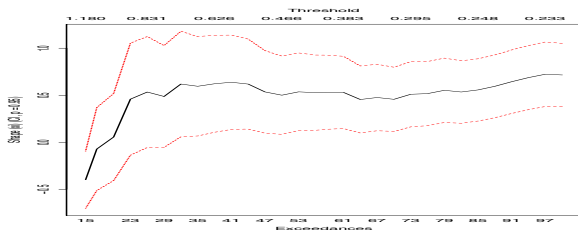
Extrapolation to high quantiles possible iff  $\xi > 0$  iff  $X \sim F$  is an  $\alpha = \xi^{-1}$  regularly varying r.v.:

## Definition (Feller, 1971)

$\exists p, q \geq 0$  with  $p + q = 1$  and a slowly varying function  $L$  such that

$$\mathbb{P}(X > x) \sim p \frac{L(x)}{x^\alpha} \quad \text{and} \quad \mathbb{P}(X \leq -x) \sim q \frac{L(x)}{x^\alpha}, \quad x \rightarrow \infty.$$

POT approach: fit by ML a GPD on the exceedances under low thresholds



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# The iid case

## Theorem (A.V. Nagaev, 1969)

$(X_i)$  iid random variables with  $\alpha > 0$  regularly varying (centered if  $\alpha > 1$ ) distribution then  $S_n = \sum_{i=1}^n X_i$  satisfies the *precise large deviations* relation

$$\lim_{n \rightarrow \infty} \sup_{x \geq b_n} \left| \frac{\mathbb{P}(S_n > x)}{n \mathbb{P}(|X| > x)} - p \right| = 0 \text{ and } \lim_{n \rightarrow \infty} \sup_{x \geq b_n} \left| \frac{\mathbb{P}(S_n \leq -x)}{n \mathbb{P}(|X| > x)} - q \right| = 0$$

with  $b_n = n^{\delta+1/(\alpha \wedge 2)}$  for any  $\delta > 0$ .

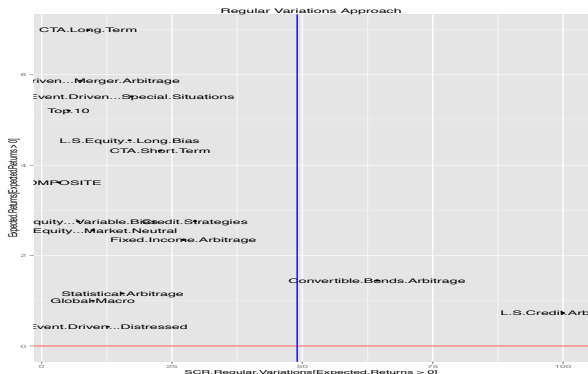
$$\text{Assume } (r_t) \text{ iid} \implies \mathbb{P}\left(\sum_{j=T}^{T+52} r_t \leq \text{SCR}\right) = 0.005 \ll \mathbb{P}(r_t \leq \text{SCR}) = 0.0001.$$

# SCR extrapolation, iid case

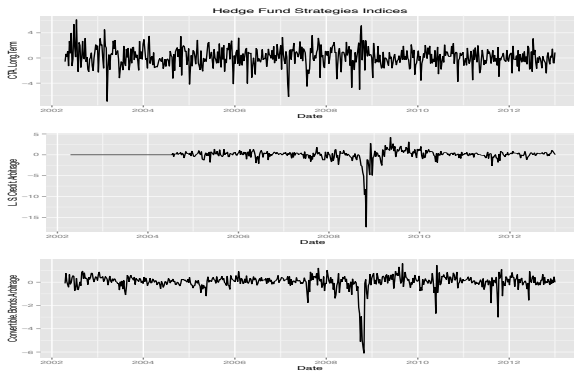
SCR with POT approach based on iid  $(r_1, \dots, r_n)$

$$\widehat{SCR} = \max_{1 \leq m \leq 40} u_m + \frac{\hat{\sigma}}{\hat{\xi}} \left[ \left( \frac{n * 0.0001}{m} \right)^{-\hat{\xi}} - 1 \right]$$

where  $m$  is the number of exceedances below  $u_m < 0$ .



# SCR extrapolation when extremes cluster



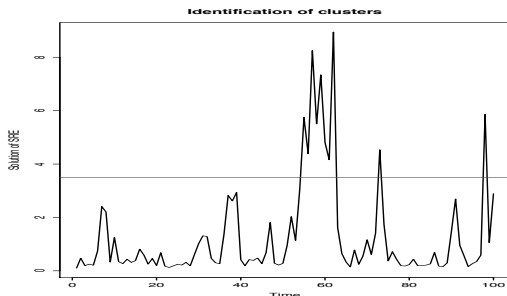
What is happening for dependent sequences for whom extremes cluster?

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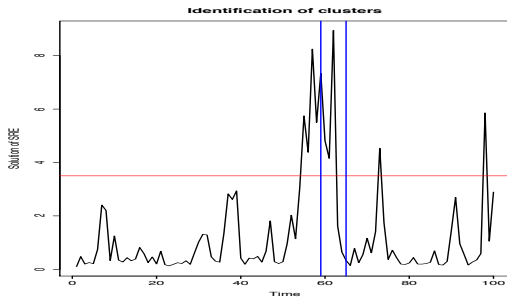
# Identification of the clusters

SRE:  $X_t = A_t X_{t-1} + B_t$ ,  $t \geq 1$  with  $(A_t, B_t)$  iid,  $A_t > 0$ ,  $\mathbb{E}A_0^\alpha = 1$  and  $\mathbb{E}|B_0|^{\alpha+\varepsilon} < \infty$ ,  $\varepsilon > 0$ . The unique stationary solution  $(X_t)$  is RV( $\alpha$ ).



How to identify the clusters?

# Approximation by local dependance (Rootzen, 1978)



When is it a good approximation when  $m \rightarrow \infty$ ?

Davis & Hsing (1995), Basrak & Segers (2009)

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \mathbb{P} \left( \max_{m \leq |i| \leq n/k_n} |X_i| > x a_n \mid |X_0| > x a_n \right) = 0, \quad x > 0. \quad (\text{ALD})$$

Under (ALD)  $\theta > 0$ , i.e. average size of clusters are finite.

# Regularly varying processes

Assume that  $(X_t, t \leq 0)$  is independent of  $(X_t, t \geq m)$  then  $\Theta_t = 0$  for  $|t| \geq m$ .

## Definition (Mikosch & W., 2013)

An  $m$ -dependent stationary sequence  $(X_t)$  is regularly varying of order  $\alpha > 0$  if  $\exists$  **conditional** spectral tail process  $(\Theta'_t)$  defined by the relation

$$\mathbb{P}(|X_0|^{-1}(X_0, \dots, X_{m-1}) \in \cdot \mid |X_0| > x, |X_{-j}| \leq \varepsilon x, 0 < j < m) \\ \xrightarrow[x \rightarrow \infty, \varepsilon \rightarrow 0]{w} \mathbb{P}((\Theta'_0, \dots, \Theta'_{m-1}) \in \cdot).$$

A significant cluster begins only if  $|X_0| > x$  and  $|X_{-j}| \leq \varepsilon x$  for  $0 < j < m$ .

# Large deviations in the $m$ -dependent case

## Theorem (Mikosch & W., 2012)

Assume  $(X_t)$  is  $\alpha > 0$  regularly varying (centered if  $\alpha > 1$ ) distribution then

$$\lim_{n \rightarrow \infty} \sup_{x \geq b_n} \left| \frac{\mathbb{P}(S_n > x)}{n \mathbb{P}(|X| > x)} - b_+ \right| = 0 \text{ and } \lim_{n \rightarrow \infty} \sup_{x \geq b_n} \left| \frac{\mathbb{P}(S_n \leq -x)}{n \mathbb{P}(|X| > x)} - b_- \right| = 0,$$

with  $b_n = n^{\delta+1/(\alpha \wedge 2)}$  for any  $\delta > 0$  and *cluster indices*

$$b_{\pm} = \gamma \mathbb{E} \left[ \left( \sum_{t=0}^{m-1} \Theta'_t \right)_{\pm}^{\alpha} \right],$$

where  $\gamma$  is the inverse of the average length of a significant cluster.

# Examples

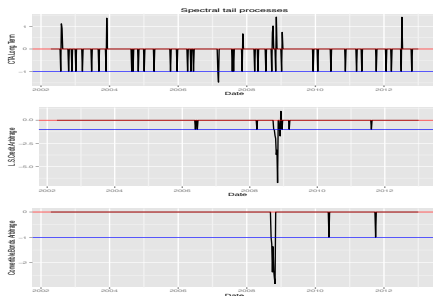
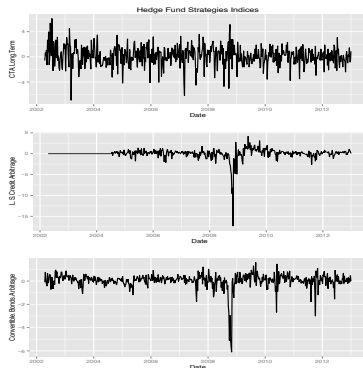
- 1  $(X_t)$  iid then  $\Theta'_0 = 1$ ,  $\Theta'_t = 0$  for  $t > 0$ ,
- 2  $X_t = \max(Z_{t-1}, Z_t)$  then  $(\Theta'_0, \Theta'_1) = (1, 1)$ ,
- 3  $X_t = Z_t + \frac{1}{2}Z_{t-1}$  then  $(\Theta'_0, \Theta'_1) = (1, \frac{1}{2})$ .

# Application to risk management

## Definition (Empirical conditional spectral tail process)

Define  $(\hat{\Theta}'_j, \dots, \hat{\Theta}'_{j+k}) = (r_j/|r_j|, \dots, r_{j+k}/|r_j|)$  if  $|r_t| > u$ ,  $t \in I = \{j, \dots, j+k\}$ ,  $|r_{j-1}| < \varepsilon u$  and  $|r_{j+k+1}| < \varepsilon u$ .

Approximation of  $(\Theta'_0, \dots, \Theta'_k)$ .



# Approximation of the cluster index

$$\mathbb{P}\left(\sum_{j=T}^{T+52} r_t \leq SCR\right) = 0.005 \approx \mathbb{P}\left(|r_t| > SCR\right) = 0.0001/b_-,$$

with  $b_- = \gamma \mathbb{E}[(\sum_{t=0}^{m-1} \Theta'_t)^{\alpha}]_-$ .

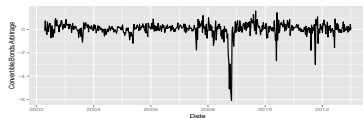
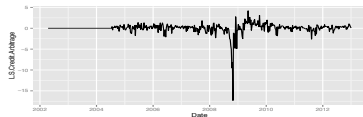
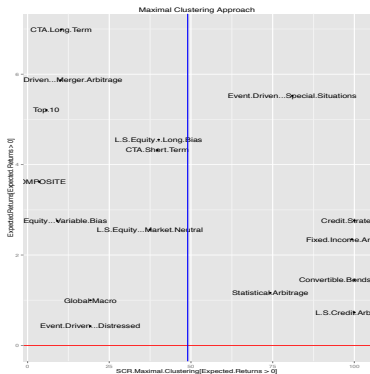
## Definition (Empirical cluster index)

$\hat{b}_-^{\ell} = \frac{1}{\ell} \sum_{i=1}^{\ell} \frac{1}{k_i} \left( \sum_{t=0}^{k_i} \hat{\Theta}'_{j_i+t} \right)_-^{\hat{\alpha}}$  where  $\ell$  is the number of significant clusters.

# Calculation of the SCR when extremes cluster

## Definition

$$\widehat{SCR} = \max_{15 \leq m \leq 40} \max_l u_m + \frac{\hat{\sigma}}{\hat{\xi}} \left[ \left( \frac{n * 0.0001}{m * \hat{b}_-^l} \right)^{-\hat{\xi}} - 1 \right].$$

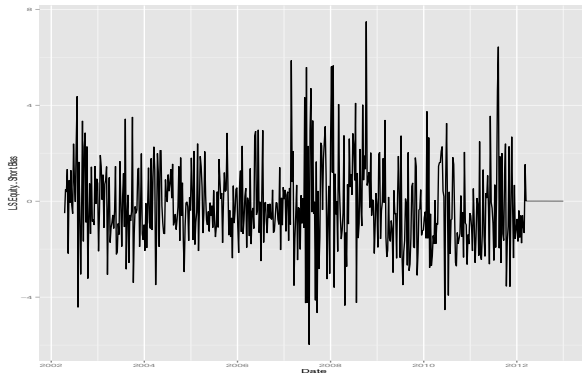


# New calculation of the SCR

Strategies	SCR.Gaussian	SCR.POT	SCR.Clustering
Convertible.Bonds	11.1	64.3	<b>100</b>
CTA.Long.Term	<b>21.6</b>	9	10.3
CTA.Short.Term	14.7	22.7	<b>39.7</b>
Event.Distressed	10.1	12.7	<b>19.2</b>
Event.Arbitrage	5.4	7.5	<b>10.1</b>
Event.Situations	13.4	17.2	<b>81</b>
Fixed.Income.Arbitrage	11.3	27.2	<b>99.2</b>
Global.Macro	16.1	9.8	<b>19.3</b>
L.S.Credit.Arbitrage	21.9	<b>100</b>	<b>100</b>
L.S.Equity.Bias	19.7	16.9	<b>40.1</b>
L.S.Equity.Neutral	6	9.8	<b>37.4</b>
L.S.Equity.Bias	<b>11.3</b>	7	9.1
Statistical.Arbitrage	12.3	15.5	<b>74.3</b>
COMPOSITE	<b>6.9</b>	3.3	3.6
Top.10	<b>9.1</b>	5.2	5.8
Credit.Strategies	8.6	29.4	<b>100</b>
Average SCR	12.5	21	<b>46.8</b>

# Conclusion

- Cluster indices  $b_{\pm}$  determine the large deviations of the sums of  $m$ -dependent regularly varying sequences,
- A new approach of risk management based on clusters of extremes to take into account the dependence.



Thank you for your attention!